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## DYNAMICS AND WELFARE IN RECOMBINANT GROWTH MODELS WITH INTELLECTUAL PROPERTY RIGHTS: A COMPUTATIONAL METHOD ¶

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# Dynamics and Welfare in Recombinant Growth Models with Intellectual Property Rights: a Computational Method

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## Abstract

We consider the extended continuous time endogenous recombinant growth model with a basic IPRs system introduced by Marchese et al. (2014). In order to analyze the effect on social welfare of different IPRs policy regimes, we must carefully study the transition dynamics associated to different values of the IPRs policy parameter. To this aim, we exploit the computational method recently developed by Privileggi (2011, 2013), based on Projection methods, Gauss-Chebyshev and Gauss-Legendre quadrature, along with standard Runge-Kutta type algorithms, to approximate such transitional dynamic paths and perform Skiba-point analysis. Our simulations show that softer IPRs policy regimes generate higher welfare levels.

**JEL Classification Numbers:** C61; C63; C68; O31; O41

**Keywords:** Knowledge Production; Endogenous Recombinant Growth; Transition Dynamics; Turnpike; Skiba Point; Welfare

## 1 Introduction

Knowledge is by far one of the most important determinants of long-run economic growth; thus, in the Economics literature great emphasis is placed on assessing the impact of different types of policies on knowledge accumulation. In such a framework, the role played by intellectual property rights (IPRs) policies is still controversial since two opposite effects need to be balanced. On the one hand, a tighter IPRs policy allows stronger incentives for economic agents to engage in knowledge creation activities; on the other hand, the same policy reduces the ability of the public domain to exploit the newly created knowledge in order to generate further innovation. The net impact of these two opposite effects determines whether tighter IPRs policy regimes might lead to higher or lower levels of social welfare, thus indicating whether they might be desirable for the society as a whole.

The main contribution of this paper is to provide a numerical approach to compute social welfare in economies endowed with some IPRs system, so to analyze the impact of different IPRs

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policy regimes on social welfare. In order to do so, our computational procedure allows to approximate the complete transition path of the continuous time, recombinant-type, endogenous growth model introduced by Marchese et al. (2014). Marchese et al. (2014) extend the Tsur and Zemel's (2007) continuous time version of the original Weitzman's (1998) growth model in order to introduce a basic IPRs system and investigate the relationship between IPRs policy and welfare in a context of recombinant knowledge creation. In such a framework knowledge does not evolve only because of profit seeking behavior (as traditionally discussed in the growth literature; see Arrow, 1962, or Romer, 1990), but also because of its autonomous progress driven by social interaction. This description of the complexity underlying knowledge accumulation is consistent with some empirical evidence (Acemoglu, 2009) and allows us to better understand the relationship between policy, growth and welfare in real world economies.

To evaluate the effect of policy on welfare, we perform a comparative dynamics exercise analyzing how different values of the IPRs policy instrument will be reflected in the evolution of consumption over time, and thus on the level of social welfare. A critical aspect of such an approach consists of computing the value of consumption along the whole transitional dynamic path. As shown in Marchese et al. (2014) and similarly in Privileggi (2010), we need to distinguish between three different kind of trajectories: those occurring along a characteristic curve labeled as 'turnpike', those outside the turnpike but eventually converging to the turnpike, and those never converging to the turnpike but ending up in stagnation. Because such types of transitional dynamics are tough objects to deal with, we rely on a wide range of numerical techniques in order to quantitatively assess different consumption paths and the social welfare they generate. The method we adopt in this paper is to a large extent built on previous works of Privileggi, who has recently developed a reliable approach to study the transitional dynamics in continuous time recombinant growth models á-la Tsur and Zemel (2007). Specifically, we first apply a suitable transformation to the ODE defining the optimal transition dynamics in order to study their associated 'detrended' system. Such transformation, although based on the same idea originally developed in Privileggi (2010), involves non trivial extensions and provide the main novel technical contribution of the paper. Next, the numerical method provided by Privileggi (2011) is applied to the detrended system in order to approximate the optimal policy along the (transitory) turnpike, while the techniques discussed in Privileggi (2013) are employed to approximate the transition dynamics outside the turnpike and toward a steady state, and to perform welfare analysis and comparative dynamics.

The paper proceeds as follows. In Section 2 we briefly recall the literature closely related to the recombinant approach to knowledge accumulation. Section 3 introduces the extended recombinant growth model (a more detailed discussion on the economic implications of the model can be found in Marchese et al., 2014) and discusses short and long-run equilibria and the eventual convergence towards an asymptotic balanced growth path equilibrium. Sections 4–6 are the core of this paper, where we present a computational approach to fully analyze all types of transition dynamics associated to different IPRs policy regimes. Specifically, in Section 4, after introducing a suitable detrendization of variables, we characterize and compute the optimal consumption path along the turnpike, while in Section 5 we focus on trajectories that start off (above) the turnpike. We develop an algorithm (based on a bisection routine) to identify the intersection point between paths starting above the turnpike and their continuation along the turnpike itself, together with the optimal policies along the early transition, so that we can build the whole optimal consumption paths as piecewise functions by joining the early trajectories with their continuation along the turnpike. In Section 6 a similar kind of analysis is performed for studying trajectories not converging to the turnpike but leading to stagnation. All these findings allow us to assess the impact of different IPRs policy regimes on welfare,

thus understanding how IPRs policy should be used in order to promote improvements in social welfare. Section 7 presents a specific illustrative example to test the performances of our approach and shows how welfare might change under three alternative IPRs policy scenarios. Section 8, as usual, concludes.

## 2 Related Literature

Weitzman (1998) firstly introduces the notion of recombinant knowledge: unprocessed ideas ('seeds' in his terminology) are combined one another in order to generate new hybrid ideas; a costly selection process allows to extract from hybrid ideas some fertile ideas which are in turn combined with other existing fertile ideas to produce new hybrid ideas, and so on. Such a recombinant process occurs indefinitely, generating knowledge accumulation and thus economic growth. The number of newly generated successful ideas in Weitzman's discrete time Solow-type (1956) framework defines a recombinant expansion process of second-order through which the stock of knowledge has the potential of growing at an increasing rate of growth. However, because the hybridization process of seed ideas necessarily also consumes an amount of physical resources, explosive growth is precluded by scarcity of resources, so that knowledge actually grows at some bounded positive rate.

Tsur and Zemel (2007) simplify the dynamics associated to the production of new ideas by reformulating the model in a continuous-time Ramsey-type (1928) framework, in which the amount of resources employed in knowledge production,  $J$ , is a variable to be optimally chosen by a benevolent social planner. In continuous-time the new knowledge production function is defined as:

$$\dot{A}(t) = H(t) \eta \left[ \frac{J(t)}{H(t)} \right], \quad (1)$$

where  $A(t)$  denotes the stock of knowledge,  $H(t)$  represents the number of hybrid seed ideas, and  $\eta(\cdot)$  is the probability of obtaining a successful idea from each hybridization (matching); specifically the success probability function  $\eta : \mathbb{R}_+ \rightarrow [0, 1]$  is increasing and concave ( $\eta' > 0$  and  $\eta'' < 0$ ) according to Weitzman's assumption (1998). Seed ideas are generated according to the following production technology:

$$H(t) = C'_m[A(t)] \dot{A}(t), \quad (2)$$

where  $C'_m(A)$  denotes the derivative of the number of different combinations of  $m$  elements as a function of the stock  $A$  (Tsur and Zemel, 2007; Privileggi, 2010, 2011). Hence, when (2) is substituted into (1), it turns out that the law of motion of knowledge at every instant is:

$$\dot{A}(t) = \frac{J(t)}{\varphi[A(t)]},$$

where  $\varphi(A) = C'_m(A) \eta^{-1} \left[ \frac{1}{C'_m(A)} \right]$  is the expected unit cost of knowledge production as a function of the stock of knowledge  $A$ , converging to the constant  $1/\eta'(0) > 0$  as  $A \rightarrow \infty$ . The knowledge-based economy in Tsur and Zemel (2007) has two types of equilibria that can be characterized by means of curves in the knowledge-capital state space: provided that a certain parametric condition is met growing economies evolve along the turnpike curve, while non growing economies converge to a point on a stagnation curve. Privileggi (2011, 2013) proposes a numerical approach to further characterize the transitional dynamics in the Tsur and Zemel's model.

While the Tsur and Zemel's model allows to describe a fully rational and perfectly informed benevolent decision's problem, Marchese et al. (2014) extend the model to allow the presence of a basic intellectual property rights system. Specifically, they assume that knowledge is a public good made publicly available for free in the social domain by the government who devotes public expenditure,  $G(t)$ , to the purchase of newly produced knowledge from R&D-firms. As a consequence, the law of motion of knowledge reads as:

$$\dot{A}(t) = \frac{G(t)}{\psi[A(t)]},$$

where  $\psi(A) = (\eta^{-1})' \left[ \frac{1}{C'_m(A)} \right]$  represents the expected price of knowledge production as a function of the stock of knowledge  $A$  paid by the government. As  $\psi(A) > \varphi(A)$  for any finite level of knowledge, firms producing knowledge end up obtaining strictly positive profits (see Proposition 1 and 2 in Marchese et al., 2014) and the government, aiming at maintaining a balanced budget at any time, can actively intervene through an appropriate tax-subsidy scheme in order to fix the inefficiency generated by these positive profits. Specifically, pure R&D profits can be taxed at a rate  $0 \leq \tau < 1$  and the proceeds can be immediately re-invested in the purchase of new R&D output. In such a framework, the law of motion of knowledge becomes:

$$\dot{A}(t) = \frac{G(t)}{\phi_\tau[A(t)]}, \quad (3)$$

with  $\phi_\tau[A(t)] = \tau\varphi[A(t)] + (1 - \tau)\psi[A(t)]$  being a linear combination (depending on the tax parameter,  $\tau$ ) of the costs of knowledge production under full centralization,  $\varphi[A(t)]$ , and under full decentralization,  $\psi[A(t)]$ . The tax parameter  $\tau$  represents the policy instrument available to the government to implement tighter or softer IPRs regimes, represented by smaller or larger values of  $\tau$  respectively. Indeed, a higher values of  $\tau$  implies a weaker appropriability of the profits generated by new knowledge creation, corresponding to a weaker protection of intellectual property (softer IPRs regimes). In the following we refer to  $\tau$  as the IPRs policy parameter. A full description of the dynamics implied by (3) and some related theoretical results can be found in Marchese et al. (2014).

### 3 Model and Asymptotic Equilibria

The model we consider is a continuous time Ramsey-type model of optimal growth with endogenous creation of knowledge, based upon the formulation presented in Marchese et al. (2014). The social planner maximizes social welfare by choosing the level of consumption,  $c(t)$ , and government expenditure,  $G(t)$ , taking into account the dynamic evolution of capital and knowledge. Social welfare is defined as the infinite discounted ( $\rho$  is the pure rate of time preference) sum of instantaneous utilities; the instantaneous utility function takes the following iso-elastic form:  $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ , where  $\sigma \geq 1$  is the inverse of the intertemporal elasticity of substitution. A unique final consumption good is competitively produced in the economy according to a Cobb-Douglas production technology combining capital,  $k(t)$ , and knowledge,  $A(t)$ :  $y(t) = F[A(t), k(t)] = \theta A(t)^{1-\alpha} k(t)^\alpha$  where  $\alpha$  is the capital share and  $\theta$  a scale parameter measuring the total factor productivity. Apart from this consumption good, in the economy also knowledge is competitively produced by R&D-firms which sell the newly created knowledge to the government, which then provides to make it freely available in the society. Output can be allocated to consumption, capital investment (for simplicity, no depreciation is assumed),  $\dot{k}(t)$ , or government spending,  $G(t)$ ; thus capital evolves according to the following law of motion:

$\dot{k}(t) = y(t) - c(t) - G(t)$ . Government expenditure is used to purchase new knowledge, which evolves according to a recombinant rule defined by (3); we assume that only pair of ideas can be matched and the probability of success in creating new ideas is described by a hyperbolic function (increasing and concave as in Weitzman, 1998), specifically  $m = 2$  and  $\eta(x) = \frac{\beta x}{\beta x + 1}$ , where  $\beta > 0$  measures the efficiency of the matching process. The last assumption implies the following expression for the unit costs of knowledge production:

$$\begin{aligned}\phi_\tau(A) &= \tau\varphi(A) + (1 - \tau)\psi(A) = \varphi(A) [(1 - \tau)\beta\varphi(A) + \tau] \\ &= \frac{1}{\beta} \left( \frac{2A - 1}{2A - 3} \right) \left[ (1 - \tau) \left( \frac{2A - 1}{2A - 3} \right) + \tau \right],\end{aligned}\quad (4)$$

which, as shown by the second equality, turns out to be a function only of the costs of knowledge production under full centralization,  $\varphi(A) = \frac{1}{\beta} \left( \frac{2A - 1}{2A - 3} \right)$ , where the last expression comes from our specification of  $\eta(\cdot)$  and the fact that  $m = 2$ .

Hence, the planner's problem can be formulated as:

$$\max_{[c(t), G(t)]_{t=t_0}^\infty} \int_0^\infty \frac{c^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t} dt \quad (5)$$

$$\text{subject to } \begin{cases} \dot{k}(t) = \theta A(t)^{1-\alpha} k(t)^\alpha - c(t) - G(t) \\ \dot{A}(t) = \frac{G(t)}{\phi_\tau(A)} \end{cases} \quad (6)$$

with the additional constraints  $G(t) \leq y(t)$ ,  $c(t) \leq k(t) + y(t)$ , the usual non-negativity constraints, and given the initial conditions  $k(0)$  and  $A(0)$ . Suppressing the time argument, the current-value Hamiltonian associated to the above problem reads as:

$$H(k, A, G, c, \lambda, \delta) = u(c) + \lambda [F(k, A) - G - c] + \delta \frac{G}{\phi_\tau(A)}, \quad (7)$$

where  $\lambda$  and  $\delta$  are the costate variables associated with  $k$  and  $A$  respectively. Along with the transversality condition and the state equations (6), first order necessary conditions are:

$$u'(c) = \lambda \quad (8)$$

$$G = \begin{cases} 0 & \text{if } \delta/\phi_\tau(A) < \lambda \\ \tilde{G} & \text{if } \delta/\phi_\tau(A) = \lambda \\ F(k, A) & \text{if } \delta/\phi_\tau(A) > \lambda \end{cases} \quad (9)$$

$$\dot{\lambda} = \rho\lambda - \lambda F_k(k, A) \quad (10)$$

$$\dot{\delta} = \rho\delta - \lambda F_A(k, A) + \delta \frac{G\phi'_\tau(A)}{\phi_\tau(A)^2}, \quad (11)$$

where  $\tilde{G}$  will be defined by (15) later.

**Remark 1** While the costates  $\lambda(t)$  and  $\delta(t)$  are continuous functions of time,<sup>1</sup> clearly conditions (9) envisage a discontinuous optimal R&D financing (a ‘bang-bang’ solution) due to linearity of the Hamiltonian in the variable  $G$ . On the other hand, through (8) and the continuity of  $u'(\cdot)$ , continuity of  $\lambda(t)$  implies that the optimal trajectory of consumption,  $c(t)$ , must be a continuous function of time as well.

<sup>1</sup>More precisely, they are *continuous and piecewise continuously differentiable* (Halkin, 1974).

As in Tsur and Zemel (2007) and Privileggi (2010, 2011, 2013), three curves in the  $(A, k)$  space are useful for characterizing the solutions of the social planner problem in our regulated economy.

1. The locus satisfying  $F_k(k, A) = \frac{F_A(k, A)}{\phi_\tau(A)}$ , which defines the (*transitory*) *turnpike* curve:

$$\tilde{k}_\tau(A) = \frac{\alpha}{1-\alpha} \phi_\tau(A) A = \frac{\alpha}{\beta(1-\alpha)} \left( \frac{2A-1}{2A-3} \right) \left[ (1-\tau) \left( \frac{2A-1}{2A-3} \right) + \tau \right] A. \quad (12)$$

2. The locus satisfying  $F_k(k, A) = \frac{F_A(k, A)}{\phi_\tau(A)}$  for large  $A$ , which defines the *asymptotic turnpike* curve:

$$\tilde{k}_\tau^\infty(A) = \frac{\alpha}{\beta(1-\alpha)} (A + 2 - \tau). \quad (13)$$

3. The locus  $F_k(k, A) = \rho$ , which defines the *stagnation line*:

$$\hat{k}(A) = (\theta\alpha/\rho)^{1/(1-\alpha)} A. \quad (14)$$

Differentiating  $\tilde{k}_\tau(A)$  in (12) with respect to time and substituting into both equations forming the dynamic constraint (6), yields

$$\tilde{G}(t) = [\tilde{y}_\tau(t) - \tilde{c}(t)] \frac{\phi_\tau[A(t)]}{\tilde{k}'_\tau[A(t)] + \phi_\tau[A(t)]}, \quad (15)$$

where  $\tilde{y}_\tau(t) = \theta \left\{ \tilde{k}_\tau[A(t)] \right\}^\alpha A(t)^{1-\alpha}$ . Condition (15) establishes a relationship between the optimal investment in R&D,  $\tilde{G}(t)$ , as a function of the other control variable, the optimal consumption  $\tilde{c}(t)$ , when the economy is constrained to grow along the curve  $\tilde{k}_\tau(A)$  defined in (12); that is, in view of (9), when  $\delta(t)/\phi_\tau[A(t)] = \lambda(t)$  holds.

As shown in Marchese et al. (2014), the economy shows positive long-run growth if the stagnation line lies above the asymptotic turnpike for sufficiently large  $A$ , that is if

$$\theta\alpha \left[ \frac{\beta(1-\alpha)}{\alpha} \right]^{1-\alpha} > \rho; \quad (16)$$

when condition (16) is not met the economy will stagnate forever (zero growth). If (16) is met, provided that the initial knowledge stock is large enough (that is,  $A_0 > \frac{3}{2}$ ) and the initial capital stock is larger than a certain threshold [that is,  $k_0 \geq k_\tau^{sk}(A_0)$ , where the superscript ‘*sk*’ is used to refer a Skiba-type (1978) point], the economy will reach the turnpike  $\tilde{k}_\tau(A)$  in a finite time, and then continue to grow along it until reaching the asymptotic turnpike  $\tilde{k}_\tau^\infty(A)$  in the long-run. In this case the economy converges towards an asymptotic balanced growth path (ABGP) characterized by a common constant growth rate of output, knowledge, capital and consumption given by:

$$\gamma = \frac{1}{\sigma} \left\{ \theta\alpha \left[ \frac{\beta(1-\alpha)}{\alpha} \right]^{1-\alpha} - \rho \right\}, \quad (17)$$

which turns out to be totally independent of the IPRs policy parameter  $\tau$ . Even if the growth rate is independent of  $\tau$ , the Skiba-type point  $k_\tau^{sk}(A_0)$ , the turnpike  $\tilde{k}_\tau(A)$  and the asymptotic turnpike  $\tilde{k}_\tau^\infty(A)$  do depend on  $\tau$ . Specifically, the turnpike curves (both the turnpike and



the asymptotic turnpike) are monotonic with respect to the IPRs policy parameter  $\tau$  (see Propositions 3 and 4 in Marchese et al., 2014).

In order to study how a given economy reacts to different IPRs policies (different values of  $0 \leq \tau < 1$ , assumed to be constant over time) we perform comparative dynamics by changing the value of parameter  $\tau$  while keeping constant all other parameters' values; specifically, we wish to characterize how the (unique) Skiba-type point  $k_\tau^{sk}(A_0)$  on the turnpike changes for different  $\tau$ -values and to compare the social welfare associated to the optimal trajectories corresponding to different IPRs policy regimes for an economy starting from the same initial pair  $(k_0, A_0)$  in  $t = 0$ . Provided that condition (16) holds and that  $k_0 \geq k_\tau^{sk}(A_0)$ , the turnpike  $\tilde{k}_\tau(A)$  is 'trapping', *i.e.*, the economy keeps growing along it after it is reached. Note that we need to distinguish between two types of transitions: one driving the system toward the turnpike starting from outside it, and another characterizing the optimal path along  $\tilde{k}_\tau(A)$  after it has been entered. Since there exists only one turnpike that crosses the initial point  $(k_0, A_0)$ , corresponding to a specific value for parameter  $\tau$ , then there is one single value for  $\tau$  such that the associated turnpike  $\tilde{k}_\tau(A)$  satisfies  $\tilde{k}_\tau(A_0) = k_0$ . Thus, in order to analyze the welfare implications of optimal trajectories corresponding to different policies starting from the same initial condition  $(k_0, A_0)$ , besides characterizing the optimal dynamics along their corresponding turnpikes, we must also study their optimal dynamics outside such turnpikes for the initial time interval required to reach them. The computational approach to study these different dynamics is presented in the next two sections, where we first numerically compute the optimal consumption  $\tilde{c}(t)$  along the turnpike  $\tilde{k}_\tau(A)$ , and then we select the unique optimal trajectory joining  $(k_0, A_0)$  (outside the turnpike) in  $t = 0$  with the turnpike itself at some instant  $t_0 > 0$  (when the turnpike is entered).

## 4 Dynamics Along the Turnpike

Let  $t_0 \geq 0$  be the instant at which the turnpike is reached [if  $k_0 = \tilde{k}_\tau(A_0)$  then  $t_0 = 0$ ]. As in Privileggi (2010, 2011, 2013), under condition (16) and assuming that  $k(t_0) \geq k_\tau^{sk}[A(t_0)]$ , for  $t \geq t_0$  the relevant variables are bound to move along the turnpike  $\tilde{k}_\tau(A)$  and the planner problem (5), (6) reduces into a simpler optimization problem in one state variable,  $A$ , and one control,  $c$ , and one dynamic constraint with reference to the initial instant  $t_0$ :<sup>2</sup>

$$\begin{aligned} \tilde{V}_\tau[A(t_0)] &= \max_{[c(t)]_{t=t_0}^\infty} \int_{t_0}^\infty \frac{c^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt \\ \text{subject to } \begin{cases} \dot{A} = [\tilde{y}_\tau(A) - c] / [\tilde{k}'_\tau(A) + \phi_\tau(A)] \\ k(t_0) = \tilde{k}_\tau[A(t_0)], \end{cases} \end{aligned} \quad (18)$$

with the additional constraint  $0 \leq c \leq \tilde{k}_\tau(A) + \tilde{y}_\tau(A)$ , where the time argument has been dropped for simplicity,  $\tilde{y}_\tau(A) = \theta \tilde{k}_\tau(A)^\alpha A^{1-\alpha}$  is the output as a function of the sole variable  $A$  on the turnpike  $\tilde{k}_\tau(A)$  as defined in (12),  $\tilde{k}'_\tau(A) = (\partial/\partial A) \tilde{k}_\tau(A)$ , and  $\phi_\tau(A)$  is given by (4). Necessary conditions on the current-value Hamiltonian for problem (18) yield the following system of ordinary differential equations (ODEs) defining the optimal dynamics for  $A$  and  $c$

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<sup>2</sup>The state variable  $k$  and the control  $G$  become functions of  $A$  and  $c$  according to (12) and (15) respectively.

along the turnpike (details are in Section 2.2 in Privileggi, 2010):

$$\begin{cases} \dot{A} = [\theta \tilde{k}_\tau(A)^\alpha A^{1-\alpha} - c] / [\tilde{k}'_\tau(A) + \phi_\tau(A)] \\ \dot{c} = c \left\{ \theta \alpha [\tilde{k}_\tau(A)/A]^{\alpha-1} - \rho \right\} / \sigma. \end{cases} \quad (19)$$

As the stock of knowledge  $A$  cannot be depleted and, since (see Proposition 3 in Marchese et al., 2014) the optimal investment in R&D must be positive along the turnpike,  $G(t) > 0$  for all  $t$ ,  $A$  must grow:  $\dot{A}(t) > 0$  for all  $t \geq t_0$ . Some characteristics of the turnpike are summarized in the next proposition.

**Proposition 1**

- i) For all  $0 < \alpha < 1$  and all  $0 \leq \tau < 1$  the graph of  $\tilde{k}_\tau(A)$  is a U-shaped curve on  $(3/2, +\infty)$ , reaching its unique minimum on a unique interior point  $\underline{A} > 3/2$ .
- ii) Moreover, for all  $0 < \alpha < 1$  and all  $0 \leq \tau < 1$  the denominator on the RHS of the first equation in (19),  $\tilde{k}'_\tau(A) + \phi_\tau(A)$ , vanishes on a unique interior point  $A^s > 3/2$ , and  $\tilde{k}'_\tau(A) + \phi_\tau(A) < 0$  for  $3/2 < A < A^s$ , while  $\tilde{k}'_\tau(A) + \phi_\tau(A) > 0$  for  $A > A^s$ ;  $A^s$  satisfies<sup>3</sup>  $3/2 < A^s < \underline{A}$ .

**Proof.** i) Differentiating  $\tilde{k}_\tau(A)$  in (12) with respect to  $A$  one gets

$$\tilde{k}'_\tau(A) = \frac{\alpha [8A^3 - 36A^2 + 2(11 + 10\tau)A - 3 - 6\tau]}{\beta(1 - \alpha)(2A - 3)^3}, \quad (20)$$

where the denominator is positive for all  $A > 3/2$ , so that the sign of  $\tilde{k}'_\tau(A)$  depends on the sign of the numerator, which is a 3<sup>rd</sup>-degree polynomial in  $A$  with positive coefficient on  $A^3$ . As its value on  $A = 3/2$  is negative, equal to  $-24(1 - \tau) < 0$ , and its derivative with respect to  $A$  is negative as well on  $A = 3/2$ , equal to  $-32 + 20\tau < 0$ , the polynomial is negative and decreasing in  $A = 3/2$ , so that there is a unique real zero  $\underline{A} > 3/2$  such that  $\tilde{k}'_\tau(\underline{A}) = 0$ , which is the unique interior minimum for  $\tilde{k}_\tau(A)$ .

ii) Using (20) and (4) we can write

$$\tilde{k}'_\tau(A) + \phi_\tau(A) = \frac{8A^3 - (20 + 16\alpha + 8\tau - 8\alpha\tau)A^2 + (14 + 8\alpha + 16\tau + 4\alpha\tau)A - 3 - 6\tau}{\beta(1 - \alpha)(2A - 3)^3}$$

where, again, the denominator is positive for all  $A > 3/2$  and the sign of  $\tilde{k}'_\tau(A) + \phi_\tau(A)$  depends on the sign of the numerator, which is a 3<sup>rd</sup>-degree polynomial in  $A$  with positive coefficient on  $A^3$ . Its value on  $A = 3/2$  is negative, equal to  $-24\alpha(1 - \tau) < 0$ , while its second derivative, equal to  $48A - 40 + 16(2 - \tau)(1 - \alpha)$ , is positive for all  $A > 3/2$ , so that the polynomial is negative on  $A = 3/2$  and convex for all  $A > 3/2$ ; therefore, there is a unique real zero  $A^s > 3/2$  such that  $\tilde{k}'_\tau(A^s) + \phi_\tau(A^s) = 0$ ,  $\tilde{k}'_\tau(A) + \phi_\tau(A) < 0$  for  $3/2 < A < A^s$  and  $\tilde{k}'_\tau(A) + \phi_\tau(A) > 0$  for  $A > A^s$ . Note that, because  $\phi_\tau(A^s) > 0$ ,  $\tilde{k}'_\tau(A^s) + \phi_\tau(A^s) = 0$  implies  $\tilde{k}'_\tau(A^s) < 0$ , and thus  $3/2 < A^s < \underline{A}$  must hold. ■

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<sup>3</sup>Points  $\underline{A}$  and  $A^s$  correspond to  $\underline{A} = 3/2 + \sqrt{6}/2$  and  $A^s = 1 + (1/2)(\alpha + \sqrt{1 + 4\alpha + \alpha^2})$  in the original Tsur and Zemel (2007) framework; they are explicitly computed in Privileggi (2010, 2011) (see Proposition 3 in Privileggi, 2010).

Proposition 1 (i) states that the graph of  $\tilde{k}_\tau(A)$  is a U-shaped curve, so that capital  $\tilde{k}(t)$  decreases when  $t$  is small and increases for larger  $t$  along the turnpike. Note that, as  $\dot{A}(t) > 0$  for all  $t \geq t_0$ , the whole ratio on the RHS of the first equation in (19) must be positive for all  $t \geq t_0$ , that is, the numerator,  $\theta \tilde{k}_\tau(A)^\alpha A^{1-\alpha} - c$ , must have the same sign of the denominator,  $\tilde{k}'_\tau(A) + \phi_\tau(A)$ , and must vanish on a unique interior point,  $A^s$ . Moreover, as  $\theta \tilde{k}_\tau(A)^\alpha A^{1-\alpha} = \tilde{y}_\tau(A)$ , Proposition 1 (ii) implies that the optimal consumption  $\tilde{c}_\tau$  must satisfy  $\tilde{c}_\tau > \tilde{y}_\tau(A)$  for  $A < A^s$ ,  $\tilde{c}_\tau < \tilde{y}_\tau(A)$  for  $A > A^s$ , and  $\tilde{c}_\tau = \tilde{y}_\tau(A)$  for  $A = A^s$ . We thus conclude that in early times it is optimal to take away physical capital from the output-producing sector both for investment in R&D and consumption.

## 4.1 Detrended Dynamics

As system (19) diverges in the long-run, we transform the state variable  $A$  and the control  $c$  in a state-like variable,  $\mu$ , and a control-like variable,  $\chi$ , defined respectively by

$$\mu = \tilde{k}_\tau(A) / A = [\alpha / (1 - \alpha)] \phi_\tau(A) \quad (21)$$

$$\chi = c / A, \quad (22)$$

where in the second equality in (21) we used (12) and  $\phi_\tau(A)$  is defined in (4). As  $\phi_\tau(\cdot)$  is strictly decreasing, there is a one-to-one relationship between  $A$  and  $\mu$ ; in order to find it explicitly, we write (21) in implicit form:

$$\beta(1 - \tau) [\varphi(A)]^2 + \tau \varphi(A) - [(1 - \alpha) / \alpha] \mu = 0, \quad (23)$$

where  $\phi_\tau(\cdot)$  has been suitably rewritten. Hence, two solutions for  $\varphi(A)$  are found, both depending on  $\mu$ ; ruling out the negative one, we can express the (first-best) unit-cost,  $\varphi$ , as a function of the detrended variable  $\mu$ :

$$\varphi(\mu) = [R(\mu) - \tau] / [2\beta(1 - \tau)], \quad (24)$$

where

$$R(\mu) = \sqrt{\tau^2 + 4[(1 - \alpha)\beta(1 - \tau) / \alpha] \mu}. \quad (25)$$

Clearly, with  $R(\mu)$  defined as in (25)  $\varphi(\mu)$  in (24) turns out to be strictly increasing in  $\mu$ . Then, using the fact that  $\varphi(A) = \frac{1}{\beta} \left( \frac{2A-1}{2A-3} \right)$ , we can write  $A$  as a strictly decreasing function of  $\varphi$ :  $A = 1 / (\beta\varphi - 1) + 3/2$ ; finally, substituting  $\varphi$  as in (24) we obtain:

$$A = 2(1 - \tau) / [R(\mu) - 2 + \tau] + 3/2, \quad (26)$$

which establishes a one-to one, strictly decreasing relationship between  $A$  and  $\mu$ .

Following the same steps as in Section 3.1 in Privileggi (2010), we divide the first equation in (19) by  $A$ , differentiate (21) with respect to time so that  $\dot{\mu} = [\tilde{k}'_\tau(A) - \mu] (\dot{A}/A)$ , substitute  $\dot{A}/A$  accordingly, and use (22) to write

$$\dot{\mu} = [\tilde{k}'_\tau(A) - \mu] (\theta \mu^\alpha - \chi) / [\tilde{k}'_\tau(A) + \phi_\tau(A)]. \quad (27)$$

In order to rewrite (27) entirely in terms of  $\mu$  and  $\chi$ , the key step is to write  $\tilde{k}'_\tau(A)$  as a function of  $\mu$ . Differentiating (12) with respect to  $A$  one gets  $\tilde{k}'_\tau(A) = [\alpha / (1 - \alpha)] [\phi'_\tau(A) A + \phi_\tau(A)]$ , so that we must compute  $\phi'_\tau(A)$  first. Differentiating  $\phi_\tau(A)$  with respect to  $A$  we get  $\phi'_\tau(A) =$

$\varphi'(A) [2\beta(1-\tau)\varphi(A) + \tau]$ ; now recall from Privileggi (2010), p. 262, that  $\varphi'(A)$  can be written as a function of  $\varphi(A)$ . In fact, differentiating and using  $\varphi(A)$ ,  $\varphi'(A) = -4/[\beta(2A-3)^2] = -(1/\beta)[2/(2A-3)]^2 = -(1/\beta)[\beta\varphi(A) - 1]^2$  holds. Thus,

$$\begin{aligned}\phi'_\tau(A) &= \varphi'(A) [2\beta(1-\tau)\varphi(A) + \tau] \\ &= -(1/\beta) [\beta\varphi(A) - 1]^2 [2\beta(1-\tau)\varphi(A) + \tau] \\ &= -\frac{1}{\beta} \left[ \frac{R(\mu) - \tau}{2(1-\tau)} - 1 \right]^2 [R(\mu) - \tau + \tau] \\ &= -\frac{R(\mu)}{\beta} \left[ \frac{R(\mu) + \tau - 2}{2(1-\tau)} \right]^2,\end{aligned}$$

where in the third equality we used (24) to replace  $\varphi(A)$  with  $\varphi(\mu)$  – the positive solution of the implicit form (23) as a function of  $\mu$  – and  $R(\mu)$  is given by (25). Substituting the expression above in  $\tilde{k}'_\tau(A) = [\alpha/(1-\alpha)][\phi'_\tau(A)A + \phi_\tau(A)]$  and using the second equality in (21),  $\phi_\tau(A) = [(1-\alpha)/\alpha]\mu$ , after a fair amount of algebra  $\tilde{k}'_\tau(A)$  becomes

$$\tilde{k}'_\tau(A) = \mu - \frac{\alpha R(\mu) [3R(\mu)^2 - 2(4-\tau)R(\mu) + 4 - \tau^2]}{8\beta(1-\alpha)(1-\tau)^2}, \quad (28)$$

which, when substituted in (27) and, again, using (21), yields

$$\begin{aligned}\dot{\mu} &= \frac{\tilde{k}'_\tau(A) - \mu}{\tilde{k}'_\tau(A) + \phi_\tau(A)} (\theta\mu^\alpha - \chi) = \left[ 1 - \frac{\mu}{\alpha\tilde{k}'_\tau(A) + (1-\alpha)\mu} \right] (\theta\mu^\alpha - \chi) \\ &= \left\{ 1 - \frac{8\beta(1-\alpha)(1-\tau)^2\mu}{8\beta(1-\alpha)(1-\tau)^2\mu - \alpha^2 R(\mu) [3R(\mu)^2 - 2(4-\tau)R(\mu) + 4 - \tau^2]} \right\} (\theta\mu^\alpha - \chi).\end{aligned}$$

Differentiating (22) with respect to time, using the second equality in (19) and substituting  $\dot{A}/A$  from the first equation in (19), we get

$$\dot{\chi} = \left[ \frac{\theta\alpha\mu^{\alpha-1} - \rho}{\sigma} - \frac{\theta\mu^\alpha - \chi}{\tilde{k}'_\tau(A) + \phi_\tau(A)} \right] \chi,$$

which, by replacing  $\tilde{k}'_\tau(A)$  as in (28) and  $\phi_\tau(A) = [(1-\alpha)/\alpha]\mu$ , yields the following ODE for the control-like variable  $\chi$ :

$$\dot{\chi} = \left[ \frac{\theta\alpha\mu^{\alpha-1} - \rho}{\sigma} - \frac{8\alpha\beta(1-\alpha)(1-\tau)^2(\theta\mu^\alpha - \chi)}{8\beta(1-\alpha)(1-\tau)^2\mu - \alpha^2 R(\mu) [3R(\mu)^2 - 2(4-\tau)R(\mu) + 4 - \tau^2]} \right] \chi.$$

Hence, we have built the following system of ODEs describing the transition optimal dynamics in the detrended variables  $\mu$  (state) and  $\chi$  (control), conjugate to the true (diverging) system (19):

$$\begin{cases} \dot{\mu} = \left[ 1 - \frac{8\beta(1-\alpha)(1-\tau)^2\mu}{Q(\mu)} \right] (\theta\mu^\alpha - \chi) \\ \dot{\chi} = \left[ \frac{\theta\alpha\mu^{\alpha-1} - \rho}{\sigma} - \frac{8\alpha\beta(1-\alpha)(1-\tau)^2(\theta\mu^\alpha - \chi)}{Q(\mu)} \right] \chi, \end{cases} \quad (29)$$

where, to simplify notation, we have set

$$Q(\mu) = 8\beta(1-\alpha)(1-\tau)^2\mu - \alpha^2 R(\mu) [3R(\mu)^2 - 2(4-\tau)R(\mu) + 4 - \tau^2], \quad (30)$$

with  $R(\mu)$  defined in (25) – *i.e.*,  $R(\mu) = \sqrt{\tau^2 + 4[\beta(1-\alpha)(1-\tau)/\alpha]}\mu$ .

Apart from the more algebraic complexity of function  $Q(\mu)$  in (30) – including  $R(\mu)$  as defined in (25) – system (29) resembles closely system (56) in Privileggi (2010) [or, equivalently, system (22) in Privileggi (2011)]. As a matter of fact, the whole qualitative behavior of the dynamics described by (29) turns out to be the same as in those models, as it will be briefly illustrated below.

## 4.2 Steady States and Phase Diagram

The state-like variable  $\mu$  has the same range as in the centralized model by Privileggi (2010, 2011, 2013):  $[\mu^*, +\infty)$  with  $\mu^* = \alpha/[\beta(1-\alpha)]$ . To see this note that  $\mu^*$  in (25) yields  $R(\mu^*) = 2 - \tau$ , which, when plugged into (26), leads to the upper bound of  $A$ ,  $A \rightarrow +\infty$ ; while, again using (25) in (26),  $\mu \rightarrow +\infty$  corresponds to  $A = 3/2$ , the lower bound of the  $A$  range.

It turns out that system (29) has *three steady states* in the  $(\mu, \chi)$  *phase diagram*, with the same properties of the three steady states characterizing the optimal dynamics for the centralized model as discussed in Privileggi (2010, 2011, 2013). From the first equation in (29) two loci on which  $\dot{\mu} = 0$  on the  $[\mu^*, +\infty) \times \mathbb{R}_{++}$  plane are found:

$$\text{the curve } \chi = \theta\mu^\alpha \text{ and the vertical line } \mu \equiv \mu^*. \quad (31)$$

The former vanishes the second factor in the RHS of the first equation in (29), while  $\mu^*$  is the largest (and only feasible) solution of  $Q(\mu) - 8\beta(1-\alpha)(1-\tau)^2\mu = 0$ , vanishing the first factor in the RHS of the same equation. By (30), such equation is equivalent to  $3R(\mu)^2 - 2(4-\tau)R(\mu) + 4 - \tau^2 = 0$ , which admits the only feasible solution<sup>4</sup>  $R(\mu^*) = 2 - \tau$ . From the second equation in (29) the unique locus on which  $\dot{\chi} = 0$  is given by:

$$\chi = \theta\mu^\alpha - Q(\mu)(\theta\alpha\mu^{\alpha-1} - \rho) / [8\alpha\beta\sigma(1-\alpha)(1-\tau)^2]. \quad (32)$$

Note that the necessary condition for growth (16) is equivalent to

$$\theta\alpha(\mu^*)^{\alpha-1} > \rho. \quad (33)$$

1. The first steady state is thus  $(\mu^*, \chi^*)$  defined by

$$\mu^* = \frac{\alpha}{\beta(1-\alpha)} \quad \text{and} \quad \chi^* = \theta \left[ \frac{\alpha}{\beta(1-\alpha)} \right]^\alpha \left( 1 - \frac{1}{\sigma} \right) + \frac{\rho}{\beta\sigma(1-\alpha)}, \quad (34)$$

where  $\chi^*$  is (32) evaluated at  $\mu = \mu^*$ , which is the intersection point between the second locus in (31),  $\mu \equiv \mu^*$ , and the curve (32). We shall see in Proposition 2 below that  $(\mu^*, \chi^*)$  is the saddle-path stable steady state to which system (29) converges in the long-run. Indeed,  $\mu^*$  corresponds to the *capital/knowledge ratio* along the asymptotic turnpike  $\tilde{k}_\tau^\infty(A)$ , *i.e.*, the slope of  $\tilde{k}_\tau^\infty(A)$  in (13), while  $\chi^*$  is the *consumption/knowledge ratio*, *i.e.*, the asymptotic slope of the optimal policy  $\tilde{c}(A)$ , when consumption steadily grows at the constant rate  $\gamma$  defined in (17). Note that  $(\mu^*, \chi^*)$  is independent of the IPRs policy parameter  $\tau$  and is the same as in Privileggi (2010, 2011, 2013).

2. As  $Q(\mu^*) - 8\beta(1-\alpha)(1-\tau)^2\mu^* = 0$  implies  $Q(\mu^*) > 0$ , (33) implies that on  $\mu = \mu^*$  the locus (32) lies strictly below the first locus in (31),  $\chi = \theta\mu^\alpha$ . However, as  $0 < \alpha < 1$ ,  $\theta\alpha\mu^{\alpha-1}$  is a decreasing function of  $\mu$ ; hence, there is a unique value  $\hat{\mu} > \mu^*$  such that

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<sup>4</sup>The other solution,  $R(\mu) = (2 + \tau)/3$ , when plugged into (26), yields  $A = 0$ , which is unfeasible.

$[\theta\alpha(\hat{\mu})^{\alpha-1} - \rho] = 0$ . It is clear from the second term in the RHS of (32) that the locus (32) and the first locus in (31) intersect in  $\mu = \hat{\mu}$ ; hence, the point  $(\hat{\mu}, \hat{\chi})$ , with coordinates

$$\hat{\mu} = (\theta\alpha/\rho)^{\frac{1}{1-\alpha}} \quad \text{and} \quad \hat{\chi} = \theta(\theta\alpha/\rho)^{\frac{\alpha}{1-\alpha}}, \quad (35)$$

is another steady state for (29). Again it is independent of  $\tau$  and equals its analogous in Privileggi (2010, 2011, 2013). From (14), it is immediately seen that the (unique) steady state  $\hat{\mu}$  in (35) corresponds to any steady state on the stagnation line  $\hat{k}(A)$  in the  $(A, k)$  space. Using  $\hat{\mu}$  into (26), we get  $\hat{A}_\tau = 2(1 - \tau) / [R(\hat{\mu}) + \tau - 2] + 3/2$ , where  $\hat{A}_\tau$  denotes the (unique) value at which the turnpike  $\tilde{k}_\tau(A)$  intersects the stagnation line  $\hat{k}(A)$  in the original model. Recalling that the expression of the analogous point for the centralized model (p. 264 in Privileggi, 2010) is  $\hat{A} = \alpha / [\beta(1 - \alpha)\hat{\mu} - \alpha] + 3/2$  and using (25) it is readily shown that  $\hat{A} < \hat{A}_\tau$  for all  $0 \leq \tau < 1$ . That is, the intersection points of the decentralized economies,  $\hat{A}_\tau$ , are all larger than the intersection point of the centralized economy,  $\hat{A}$ .

3. The last steady state corresponds to the second intersection point between the locus (32) and the first locus in (31), which is identified by a zero of function  $Q(\mu)$ , again vanishing the second term in the RHS of (32). As  $R(\mu)$  in 25 is strictly increasing in  $\mu$ , it is invertible, so that  $\mu = \alpha(R^2 - \tau^2) / [4\beta(1 - \alpha)(1 - \tau)]$ ; after substituting into (30), we can rewrite  $Q(\mu)$  as a  $3^{rd}$ -degree polynomial in  $R$  with negative coefficient on the term  $R^3$ :  $\overline{Q}(R) = -3\alpha^2 R^3 + 2\alpha[1 + 4\alpha - (1 + \alpha)\tau]R^2 - \alpha^2(4 - \tau^2)R - 2\alpha(1 - \tau)\tau^2$ . Recall that  $R^* = R(\mu^*) = 2 - \tau$  is the left endpoint of the range for  $R$ ,  $[R^*, +\infty)$ , corresponding to the range  $[\mu^*, +\infty)$  for  $\mu$ . As  $Q(\mu^*) = \overline{Q}(R^*) > 0$  and its derivative is positive as well on  $R^*$ ,  $\overline{Q}'(R^*) = 4(1 - \alpha)(\tau^2 - 3\tau + 2) > 0$  for all  $0 \leq \tau < 1$ , there is a unique feasible real zero, call it  $R^s > R^*$ , such that  $\overline{Q}(R^s) = 0$ ,  $\overline{Q}(R) > 0$  for  $R^* \leq R < R^s$  and  $\overline{Q}(R) < 0$  for  $R > R^s$ . To  $R^s$  corresponds  $\mu_\tau^s$  which, according to (26), defines our third and last steady state  $(\mu_\tau^s, \chi_\tau^s)$  as

$$\mu_\tau^s = \alpha[(R^s)^2 - \tau^2] / [4\beta(1 - \alpha)(1 - \tau)] \quad \text{and} \quad \chi_\tau^s = \theta(\mu_\tau^s)^\alpha. \quad (36)$$

Note that  $\mu_\tau^s$  is defined as a function of  $R^s$ , which cannot be explicitly computed as function of parameters  $\alpha$  and  $\tau$ ; however, for our purposes it will be sufficient to calculate it whenever some values for  $\alpha$  and  $\tau$  are chosen. Thus,  $\mu_\tau^s > \mu^*$  is the largest (and only admissible) zero of the function  $Q(\mu)$  defined in (30), with  $Q(\mu) > 0$  for  $\mu^* \leq \mu < \mu_\tau^s$  and  $Q(\mu) < 0$  for  $\mu > \mu_\tau^s$ . The value  $\mu_\tau^s$  corresponds to the critical point  $A^s$  on which  $c = \tilde{y}_\tau(A^s)$  in the  $(A, c)$  space discussed in Proposition 1 (ii). The steady state  $(\mu_\tau^s, \chi_\tau^s)$  defined in (36) is the only one depending on the IPRs policy parameter  $\tau$  in our model; in other words, it is the only steady state that differs from its analogous defined by (62) on p. 264 in Privileggi (2010).

While the singular point  $(\mu_\tau^s, \chi_\tau^s)$  lies north-east of the long-run steady state  $(\mu^*, \chi^*)$  for all admissible parameters' values, the position of  $(\hat{\mu}, \hat{\chi})$  depends on the magnitude of the discount factor  $\rho$  with respect to parameters  $\alpha$ ,  $\theta$  and  $\beta$ . Following Privileggi (2010, 2011, 2013), we shall assume that

$$\theta\alpha(\mu_\tau^s)^{\alpha-1} < \rho < \theta\alpha(\mu^*)^{\alpha-1}, \quad (37)$$

envisaging a phase diagram in which  $(\hat{\mu}, \hat{\chi})$  lies north-east of  $(\mu^*, \chi^*)$  and south-west of  $(\mu_\tau^s, \chi_\tau^s)$ . Note that the RHS in (37) equals the necessary condition (33) for long-run growth.

**Proposition 2** *Provided that (37) holds, the two fixed points  $(\mu^*, \chi^*)$  and  $(\hat{\mu}, \hat{\chi})$ , defined in (34) and (35) respectively, can be classified as follows.*

- i)  $(\mu^*, \chi^*)$  is saddle-path stable, with the stable arm converging to it from north-east whenever the initial values  $(\mu(t_0), \chi(t_0))$  are suitably chosen.
- ii)  $(\hat{\mu}, \hat{\chi})$ , with coordinates defined in (35), is an unstable clockwise-rotating spiral.

**Proof.** By studying both the phase diagram and/or the Jacobian of (29) evaluated at  $(\mu^*, \chi^*)$  and  $(\hat{\mu}, \hat{\chi})$  the result is readily shown; we omit the cumbersome calculations for brevity. As a matter of fact, the Jacobian on  $(\mu^*, \chi^*)$  turns out to be the same as that of the centralized model; also the Jacobian on  $(\hat{\mu}, \hat{\chi})$  is the same as its counterpart in the centralized model, with the appropriate  $Q(\hat{\mu})$  as in (30) in place of that defined in (57) on p. 263 in Privileggi (2010). We refer the reader to the proof of Proposition 4 in Privileggi (2010) for details. ■

Like in the centralized model, the critical point  $(\mu_\tau^s, \chi_\tau^s)$  defined in (36) is a ‘supersingular’ steady state whose Jacobian contains elements diverging to infinity, so that its stability/instability properties cannot be classified analytically. It corresponds to the value  $A^s$  on which the RHS of the first equation in (19) is not defined (both the numerator and the denominator vanish) as discussed in Proposition 1 (ii). It is crossed by the stable arm of the saddle-path at low values of the stock of knowledge  $A$ , that is, in early times (in proximity of the very beginning of the economy’s dynamics). Proposition 2 (ii) implies that, like in the centralized model, the steady state  $(\hat{\mu}, \hat{\chi})$  is irrelevant for our analysis, as the optimal trajectory keeps well apart from it. The qualitative phase diagram associated to (29) is the same as that in Figure 1 on p. 266 in Privileggi (2010), here reported in Figure 1, where all loci are drawn and stability/instability properties of the three steady states are illustrated. We refer the reader to that paper for a more thoroughly discussion of the qualitative properties of the three steady states  $(\mu^*, \chi^*)$ ,  $(\hat{\mu}, \hat{\chi})$  and  $(\mu_\tau^s, \chi_\tau^s)$ .

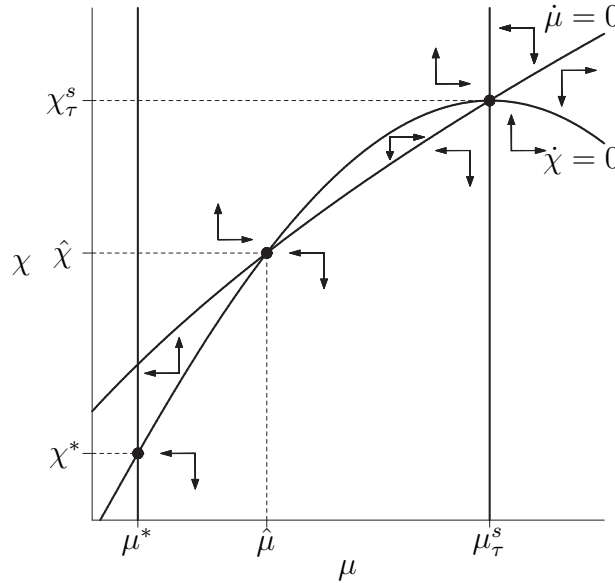


FIGURE 1: phase diagram of system (29) when  $\theta\alpha(\mu_\tau^s)^{\alpha-1} < \rho < \theta\alpha(\mu^*)^{\alpha-1}$ .

### 4.3 Optimal Policy, Welfare and Time-path Trajectories

Remember that  $\tilde{k}_\tau(A) > \tilde{k}_\tau^\infty(A)$  for all  $A$  (and thus for all  $t$ ); this is consistent with  $\mu(t) > \mu^*$  for all  $t$ . Because, by Proposition 2 (i), the stable arm  $\chi(\mu)$  – which is the *optimal policy*

expressed in terms of state-like and control-like variables – approaches  $(\mu^*, \chi^*)$  from north-east, along the turnpike both ratios  $\mu(t) = \tilde{k}_\tau[A(t)]/A(t)$  and  $\chi(t) = \tilde{c}(t)/A(t)$  decline in time when they are approaching the asymptotic turnpike  $k_\tau^\infty(A)$  corresponding to  $(\mu^*, \chi^*)$ .

In order to study the policy function  $\chi(\mu)$  – which is the conjugate of  $\tilde{c}(A)$  in the original model – we apply the technique developed by Mulligan and Sala-i-Martin (1991) and tackle the unique ODE given by the ratio between the two equations in (29):

$$\chi'(\mu) = \frac{[(\alpha\theta\mu^{\alpha-1} - \rho)/\sigma] Q(\mu) - 8\alpha\beta(1-\alpha)(1-\tau)^2 [\theta\mu^\alpha - \chi(\mu)]}{[Q(\mu) - 8\beta(1-\alpha)(1-\tau)^2 \mu] [\theta\mu^\alpha - \chi(\mu)]} \chi(\mu), \quad (38)$$

where  $Q(\mu)$  is defined in (30). Following Privileggi (2011, 2013), to numerically approximate the solution,  $\chi_\tau(\mu)$ , of (38) we apply a *Projection method* based on *OLS* of the residual function associated to equation (38), assuming that the approximate solution is a linear combination of *Chebyshev polynomials* up to *degree n* translated on an interval  $[\underline{\mu}_\tau, \bar{\mu}_\tau]$  – whose endpoints  $\underline{\mu}_\tau$  and  $\bar{\mu}_\tau$  and further details will be discussed in Subsection 7.1 – containing the two relevant steady states,  $(\mu^*, \chi^*)$  and  $(\mu_\tau^s, \chi_\tau^s)$ , which are used to set the initial condition for the OLS algorithm. Such function  $\chi_\tau(\mu)$  is then used together with (21) and (22) to get the *optimal consumption policies* for problem (18) according to

$$\tilde{c}_\tau(A) = \chi_\tau(\mu) A = \chi_\tau \left[ \frac{\alpha}{\beta(1-\alpha)} \phi_\tau(A) \right] A, \quad (39)$$

where  $\phi_\tau(A)$  is defined in (4).

For parameter values [see (51) in Section 7] satisfying condition (48) of Proposition 4 in Privileggi (2013), we can compute the value functions associated to (18) – yielding social welfare along the turnpikes and toward the ABGP as functions of the initial stock of knowledge  $A$  and independently of the starting instant  $t_0$ , for any  $\tau$ -value in  $[0, 1)$  – by means of the Hamilton-Jacobi-Bellman (HJB) equation as [see equation (47) in Privileggi, 2013]

$$\tilde{V}_\tau(A) = \frac{1}{\rho} \left[ \frac{\tilde{c}_\tau(A)^{1-\sigma} - 1}{1-\sigma} + \frac{\theta \tilde{k}_\tau(A)^\alpha A^{1-\alpha} - \tilde{c}_\tau(A)}{\tilde{c}_\tau(A)^\sigma} \right], \quad (40)$$

where  $\tilde{c}_\tau(A)$  is given by (39).

To get the time-path trajectories  $\mu_\tau(t)$  for any fixed value  $\tau \in [0, 1)$  we substitute the approximated optimal policies  $\chi_\tau(\mu)$  into the first equation of (29), which is then numerically solved through the standard *Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant* available in Maple 16, using the initial conditions given by the upper bounds  $\bar{\mu}_\tau$ . The time-path trajectories for the detrended controls are thus computed as  $\chi_\tau[\mu_\tau(t)]$ . The time-path trajectories of the stock of knowledge,  $\tilde{A}_\tau(t)$ , and capital,  $\tilde{k}_\tau(t)$ , along the turnpikes are obtained using  $\mu_\tau(t)$  in (26) and then computing  $\tilde{k}_\tau(t) = \tilde{k}_\tau[\tilde{A}_\tau(t)]$  from the definition of turnpike in (12). Similarly, the time-path trajectories of output along the turnpikes are given by  $\tilde{y}_\tau(t) = \theta \tilde{A}_\tau(t) \left\{ \tilde{k}_\tau[\tilde{A}_\tau(t)] / \tilde{A}_\tau(t) \right\}^\alpha$ , while the time-path trajectories of the optimal consumption along the turnpikes,  $\tilde{c}_\tau(t)$ , are obtained using trajectories  $\chi_\tau(t)$  and  $\tilde{A}_\tau(t)$  in (22).

## 5 Dynamics and Welfare Above the Turnpike

We noted at the end of Section 3 that also optimal trajectories starting ‘outside’ the turnpike at  $t = 0$  and entering the turnpike at some later instant  $t_0 > 0$  must be considered. As observed



in Privileggi (2013), trajectories starting from initial conditions  $(A_0, k_0)$  that lie ‘above’ the turnpike at  $t = 0$  are easier to handle; therefore, we assume that  $k(0) = k_0 \geq \tilde{k}_\tau[A(0)] = \tilde{k}_\tau(A_0)$ . When  $k_0 > \tilde{k}_\tau(A_0)$  we must study the optimal trajectories from  $(A_0, k_0)$  in  $t = 0$  to  $(A(t_0), \tilde{k}_\tau[A(t_0)])$  in  $t = t_0 > 0$ . Any optimal trajectory above the turnpike must satisfy the last necessary condition in (9),  $\delta/\phi_\tau(A) > \lambda$ , corresponding to the largest possible investment in R&D activities by the government:<sup>5</sup>  $G = \theta k^\alpha A^{1-\alpha}$ . In other words, from the social planner point of view it is optimal to invest all the output into the production of new knowledge along such early-transition trajectories. Hence, problem (5), (6) simplifies into one in two interlinked state variables,  $A$  and  $k$ , and one control variable,  $c$ :

$$\begin{aligned} \tilde{V}_\tau^{ab}[A_0, A(t_0)] &= \max_{[c(t)]_{t=0}^{t_0}} \int_0^{t_0} \frac{c^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t} dt \\ \text{subject to } &\begin{cases} \dot{k} = -c \\ \dot{A} = \theta k^\alpha A^{1-\alpha} / \phi_\tau(A) \\ k(0) = k_0, A(0) = A_0 \\ k(t_0) = \tilde{k}_\tau[A(t_0)], c(t_0) = \tilde{c}_\tau[A(t_0)], \end{cases} \end{aligned} \quad (41)$$

where the superscript ‘ab’ is used to refer any function related to dynamics *above* the turnpike, with the additional constraint  $0 \leq c \leq k$ , where again the time argument has been dropped for simplicity. The terminal conditions  $k(t_0) = \tilde{k}_\tau[A(t_0)]$  and  $c(t_0) = \tilde{c}_\tau[A(t_0)]$  bound our trajectory to reach the turnpike  $\tilde{k}_\tau(A)$  at time  $t = t_0$  with the same consumption value as the optimal consumption  $\tilde{c}_\tau[A(t_0)]$  on the turnpike evaluated according to (39) on  $A = A(t_0)$ . The latter condition holds because the control  $c$  – the optimal consumption – is continuous for all  $t \geq 0$ , as noted in Remark 1.

Necessary conditions on the current-value Hamiltonian associated to (41) yield the following optimal dynamics:

$$\begin{cases} \dot{k} = -c \\ \dot{A} = \theta k^\alpha A^{1-\alpha} / \phi_\tau(A) \\ \dot{c} = c [\theta \alpha (k/A)^{\alpha-1} - \rho] / \sigma, \end{cases} \quad (42)$$

which is a boundary problem in three variables,  $k$ ,  $A$ ,  $c$ , and one unknown, the terminal instant at which the turnpike is entered,  $t_0$ , with four boundary conditions – the initial and terminal conditions in problem (41).

To approximate the solution of (41) we take the ratios  $\dot{k}/\dot{A}$  and  $\dot{c}/\dot{A}$  in system (42) and study the following system of ODEs in the functions  $k(A)$  and  $c(A)$  (see Mulligan and Sala-i-Martin, 1991):

$$\begin{cases} k'(A) = -\frac{c(A) \phi_\tau(A)}{\theta k(A)^\alpha A^{1-\alpha}} \\ c'(A) = \frac{\{\theta \alpha [k(A)/A]^{\alpha-1} - \rho\} c(A) \phi_\tau(A)}{\theta k(A)^\alpha A^{1-\alpha}}. \end{cases} \quad (43)$$

Provided that we know the value  $A_\tau^* = A(t_0)$  on which at instant  $t_0$  the trajectories starting from  $(A_0, k_0)$  in  $t = 0$  and defined by the dynamics above stop and switch regime becoming the optimal dynamics along the turnpike  $\tilde{k}_\tau(A)$  discussed in the previous subsection, to solve (43) we apply a *Projection method* based on *Chebyshev Orthogonal Collocation* on  $n$  collocation points over the (compact) interval  $[A_0, A_\tau^*]$  applied to the two residual functions – one for each policy  $k_\tau^{ab}(A)$  and  $c_\tau^{ab}(A)$  to be estimated – built upon approximation functions which are linear

<sup>5</sup>See Proposition 1 on p. 3464 in Tsur and Zemel (2007).

combinations of  $n$  Chebyshev polynomials. As initial condition for the *Maple 16 'fsolve' routine* used to numerically solve the system of  $2n + 2$  equations setting the two residual functions equal to zero on each collocation node plus the two terminal conditions  $k_\tau^{ab}(A) = \tilde{k}_\tau(A_r)$  and  $c_\tau^{ab}(A) = \tilde{c}_\tau(A_r)$  available from the calculations in the previous subsection, we use a *Chebyshev regression of order  $n$*  on the lines crossing the pairs of points  $(A_0, \tilde{k}_\tau(A_0))$ ,  $(A_r, \tilde{k}_\tau(A_r))$  and  $(A_0, \tilde{c}_\tau(A_0))$ ,  $(A_r, \tilde{c}_\tau(A_r))$  for the  $k_\tau^{ab}(A)$  and  $c_\tau^{ab}(A)$  policies respectively. Further details can be found in Section 5 of Privileggi (2013).

Note that the terminal conditions  $k_\tau^{ab}(A) = \tilde{k}_\tau(A_r)$  and  $c_\tau^{ab}(A) = \tilde{c}_\tau(A_r)$  – and thus the whole policies  $k_\tau^{ab}(A)$  and  $c_\tau^{ab}(A)$  over  $[A_0, A_r^r]$  – crucially depend on the number  $A_r^r$ . As a matter of fact, both the value  $A_r^r$  and the instant  $t_0$  at which  $A_r^r$  is reached are unknown. The following Algorithm 1, based on a bisection routine, aims at finding  $A_r^r$  by searching the unique zero of the function

$$f(A_r^r) = k_\tau^{ab}(A_0) - k_0, \quad (44)$$

where  $k_\tau^{ab}(A_0)$  is the initial capital level corresponding to  $A_0$  at  $t = 0$  along the backward-in-time trajectory from the point  $(A_r^r, \tilde{k}_\tau(A_r^r))$  on the  $\tau$ -valued turnpike according to the capital optimal policy  $k_\tau^{ab}(A)$  solving (43), and  $k_0$  is the initial capital value set in problem (41). In other words, the algorithm runs several estimates of the solution of system (43) – corresponding to different intersection points  $(A_r^r, \tilde{k}_\tau(A_r^r))$  – and each estimated policy  $k_\tau^{ab}(A)$  is then evaluated (backward-in-time) at the initial knowledge value  $A_0$  until it matches the true initial capital value  $k_0$ . Besides yielding  $A_r^r$ , clearly Algorithm 1 provides also the associated optimal policies  $k_\tau^{ab}(A)$  and  $c_\tau^{ab}(A)$  over  $[A_0, A_r^r]$  as its output.

Because, by construction, we are considering initial capital endowments that lie ‘above’ the turnpike at  $t = 0$ ,  $k_0 > \tilde{k}_\tau(A_0)$ , so that  $f(A_0) < 0$ ,  $A_0$  clearly plays a useful role as left endpoint of the initial interval bracketing the unique zero of  $f$  in (44). As far as the right endpoint of such bracket is concerned, we shall see from the analysis in Subsection 7.2 (see Table 1) that the knowledge value corresponding to the Skiba-point on each turnpike considered in Section 7 turns out to be larger than our choice of initial stock of knowledge,  $\tilde{A}_\tau^{sk} > A_0$ , for the  $\tau$ -values there considered. Therefore, we can start from the known value  $\tilde{A}_\tau^{sk}$  and then add subsequent increments until a knowledge point  $A_R$  is found such that  $f(A_R) > 0$ , then we set  $[A_0, A_R]$  as the initial bracketing interval for the zero of  $f$  we are looking for.

**Algorithm 1 (Finds intersection point  $A_r^r$  and policies  $k_\tau^{ab}(A)$ ,  $c_\tau^{ab}(A)$  over  $[A_0, A_r^r]$ )**

**Step 1 (Initialization):** Choose  $0 \leq \tau < 1$  and set  $[A_L, A_R] = [A_0, \tilde{A}_\tau^{sk}]$ , with  $\tilde{A}_\tau^{sk}$  being the knowledge stock corresponding to the (unique) Skiba-point on the turnpike  $\tilde{k}_\tau(A)$ , as the initial interval for searching the interval bracketing the zero of  $f$  in (44), set a (switch) variable  $B = 1$ , choose an increment  $\epsilon > 0$ , choose stopping rule parameters  $0 < \epsilon, \eta < 1$ , and set (fake) initial values  $f(A_r^r) = f(A_R) = 1 > \eta$ .

**Step 2 (Bisection Loop):** While  $A_R - A_L > \epsilon$  and  $|f(A_r^r)| > \eta$  do:

1. if  $B = 1$  then set  $A_R = A_R + \epsilon$  (increase right bound) and  $A_r^r = A_R$ , else set  $A_r^r = (A_R + A_L) / 2$  (compute midpoint),
2. approximate policies  $k_\tau^{ab}(A)$  and  $c_\tau^{ab}(A)$  over  $[A_0, A_r^r]$  by solving (43) through the Collocation-Projection method described above,
3. compute  $k_\tau^{ab}(A_0)$  evaluating the policy  $k_\tau^{ab}(A)$  found in the previous step on  $A = A_0$ ,

4. update  $f(A_\tau^r)$  in (44) by setting  $f(A_\tau^r) = k_\tau^{ab}(A_0) - k_0$ ,
5. if  $B = 1$  and  $f(A_\tau^r) < 0$  then (keep searching for bracket right endpoint) go to step 2, else (bisection loop)
  - if  $B = 1$  set  $B = 0$  (stop searching for bracket),
  - refine the bounds: if  $f(A_\tau^r) f(A_R) < 0$  then set  $A_L = A_r$ , else set  $A_R = A_r$  and update  $f(A_R)$  by setting  $f(A_R) = f(A_\tau^r)$ .

**Step 3:** Report the intersection point value  $A_\tau^r$  from step 2.1 and optimal policies  $k_\tau^{ab}(A)$  and  $c_\tau^{ab}(A)$  over  $[A_0, A_\tau^r]$  from step 2.2.

Once we have the functions  $k_\tau^{ab}(A)$  and  $c_\tau^{ab}(A)$  over  $[A_0, A_\tau^r]$  we can numerically compute the optimal time-path trajectories  $A_\tau^{ab}(t)$ ,  $k_\tau^{ab}(t)$ ,  $y_\tau^{ab}(t)$  and  $c_\tau^{ab}(t)$  between  $t = 0$  and the instant  $t_0$  at which each turnpike is reached. To get the optimal time-path trajectory of the stock of knowledge  $A_\tau^{ab}(t)$  along this early transition dynamic for the economy,  $k_\tau^{ab}(A)$  is substituted into the first equation of (42) so to obtain a ODE with respect to time which can be numerically solved through the standard *Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant* method available in Maple 16, using the initial condition  $A_\tau^{ab}(0) = A_0$  in  $t = 0$  over a tentative time range of  $[0, t_{\max}]$ , with the right endpoint larger than the intersection instant  $t_0$ , that is, such that  $t_{\max} > t_0$ . The choice of  $t_{\max}$  comes from a guess-and-try approach and is delicate issue as it must be larger than  $t_0$  – which is still unknown – but not too large to prevent the Maple 16 Runge-Kutta algorithm to collapse (see also Remark 4 in Privileggi, 2013). Next, we eventually calculate  $t_0$  by solving  $A_\tau^{ab}(t) = A_r$  with respect to  $t$  through the Maple 16 ‘*fsolve*’ routine over  $[0, t_{\max}]$ . The other optimal time-path trajectories over  $[0, t_0]$  are then computed as  $k_\tau^{ab}(t) = k_\tau^{ab}[A_\tau^{ab}(t)]$ ,  $y_\tau^{ab}(t) = G_\tau^{ab}(t) = \theta k_\tau^{ab}(t)^\alpha A_\tau^{ab}(t)^{1-\alpha}$  and  $c_\tau^{ab}(t) = c_\tau^{ab}[A_\tau^{ab}(t)]$  respectively.

The whole optimal time-path trajectories,  $A_\tau(t)$ ,  $k_\tau(t)$ ,  $y_\tau(t)$ ,  $c_\tau(t)$  and  $G_\tau(t)$ , for all  $t \geq 0$  are then built as *piecewise functions* by joining each trajectory above the turnpike over  $[0, t_0]$  with its ‘continuation’ along the turnpike over  $(t_0, +\infty)$  at the instant  $t = t_0$ , the latter being obtained through the procedure discussed at the end of Subsection 4.3 and shifted forward in time up to the instant at which the knowledge stock value  $A_\tau^r$  is reached. As  $G(t) = y(t)$  for  $0 \leq t < t_0$  while  $G(t) = \tilde{G}(t) < y(t)$  for  $t \geq t_0$ , with  $\tilde{G}(t)$  given by (15), we expect to observe a discontinuity ‘jump’ for the control  $G$  at the instant  $t = t_0$  [see necessary conditions (9)], while all other trajectories must exhibit a kink on  $t = t_0$ , where they are not differentiable (see Figures 6 and 7 in Section 7).

We shall see in Subsection 7.3 that the speed of the economy’s growth is heavily affected by whether the initial condition  $(A_0, k_0)$  lies either ‘on’ or ‘above’ the turnpike at  $t = 0$ . Because, when  $k$  lies above the turnpike, for the social planner it is optimal to invest all output in R&D activities, when  $k_0 > \tilde{k}_\tau(A_0)$  the time period,  $t_0$ , required to reach the turnpike is much shorter than the time period necessary to reach the same point on the turnpike when starting already on the turnpike, along which it is optimal to invest only a fraction of the output in R&D activities.

Finally, to estimate the social welfare associated to the whole piecewise-built time-path trajectory toward the ABGP starting at  $t = 0$  from  $(A_0, k_0)$ , we apply the Principle of Optimality and again Proposition 4 in Privileggi (2013). Specifically, after having computed the intersection instant  $t_0$ , we can exploit the early optimal consumption trajectory  $c_\tau^{ab}(t)$  just calculated and conveniently define the welfare as the sum of two terms:

$$\tilde{V}_\tau^{ab}(A_0, A_\tau^r) = \int_0^{t_0} \frac{e^{-\rho t} c_\tau^{ab}(t)^{1-\sigma} - 1}{1-\sigma} dt + e^{-\rho t_0} \tilde{V}_\tau(A_\tau^r), \quad (45)$$

where  $\tilde{V}_\tau(A_\tau^r)$  is the value function of problem (18) according to (40) evaluated at the intersection point  $A_\tau^r$  obtained from Algorithm 1. That is, at  $t = t_0$  we consider the welfare generated along the turnpike when it starts with initial stock of knowledge  $A_\tau^r$ , and discount this value in  $t = 0$ . The first integral on the RHS of (45) is approximated through a *Gauss-Legendre quadrature* routine on a large number of nodes over the time range  $[0, t_0]$ , using the time-path trajectory value of optimal consumption,  $c_\tau^{ab}(t)$ , defined before on each node.

## 6 Dynamics and Welfare Toward Stagnation

Finally, we must also evaluate social welfare toward the steady state  $(A_0, \hat{k}(A_0))$  on the stagnation line starting from the initial point  $(A_0, k_0)$  in  $t = 0$ . This value can be compared with those obtained from trajectories leading to the ABGP as results of the analysis in Subsection 4.3 and Section 5 in order to check whether the Skiba condition is satisfied. According to condition  $\delta/\phi_\tau(A) < \lambda$  in (9), the optimal dynamics characterizing this scenario follow a constant zero-R&D investment policy,  $G(t) \equiv 0$ , and are just standard saddle-path stable trajectories of a typical Ramsey model in which the level  $A_0$  of knowledge stock remains constant through time. That is, they are solutions of the following problem in the two variables  $k$  (state) and  $c$  (control), and the usual dynamic constraint:

$$\begin{aligned} \bar{V}(A_0, k_0) = \max_{[c(t)]_{t=0}^{t_0}} & \int_0^\infty \frac{c^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t} dt \\ \text{subject to } & \begin{cases} \dot{k} = \theta k^\alpha A_0^{1-\alpha} - c, \\ k(0) = k_0. \end{cases} \end{aligned} \quad (46)$$

Necessary conditions on the current-value Hamiltonian associated to (46) easily lead to the following well-known conditions:

$$\begin{cases} \dot{k} = \theta k^\alpha A_0^{1-\alpha} - c \\ \dot{c} = c [\theta \alpha (k/A_0)^{\alpha-1} - \rho] / \sigma. \end{cases} \quad (47)$$

Rescaling the variables  $k$  and  $c$  in system (47) by the ratios  $\mu = k/A_0$  and  $\chi = c/A_0$  and keeping  $A_0$  constant, we can evaluate the optimal policy associated to (46) in the same ‘detrended’  $(\mu, \chi)$  space that contains the optimal policy  $\tilde{\chi}_\tau(\mu)$  of model (18) given by the solution of (38) obtained in Subsection 4.3. A constant stock of knowledge  $A \equiv A_0$  implies  $\dot{A} \equiv 0$ , which allows to rewrite system (47) as

$$\begin{cases} \dot{\mu} = \theta \mu^\alpha - \chi \\ \dot{\chi} = \chi (\theta \alpha \mu^{\alpha-1} - \rho) / \sigma. \end{cases}$$

Again we take the ratio of equations above and study the unique ODE characterizing the optimal policy in this scenario:

$$\chi'(\mu) = \frac{(\theta \alpha \mu^{\alpha-1} - \rho) \chi(\mu)}{\sigma [\theta \mu^\alpha - \chi(\mu)]}. \quad (48)$$

Following Privileggi (2013) we approximate the solution of (48) through a *Projection method* based on *Chebyshev Orthogonal Collocation* on  $n$  collocation points applied to a residual function built upon an approximation function which is a linear combination of  $n$  Chebyshev polynomials translated over the (compact) interval  $[\hat{\mu}, \mu_{\max}]$ , where  $\hat{\mu} = \hat{k}(A)/A$  is the steady state defined in (35), which happens to be neutral with respect to the optimal dynamics on the

turnpike discussed in Section 4, representing all points on the stagnation line in the detrended space<sup>6</sup>  $(\mu, \chi)$ , for all  $A$ , while  $\mu_{\max}$  is the upper-bound choice for the (global) analysis along the turnpike that will be discussed in Subsection 7.1. Such choice for the right endpoint  $\mu_{\max}$  of the range over which a solution of (48) is being approximated allows for a direct comparison between trajectories that start on or above the turnpike and either diverge toward the ABGP or converge to stagnation, as will become clear in Subsection 7.2. See also footnote 10 in Privileggi (2013) for further technical details.

The *optimal consumption policy* for problem (46), which is a function of the only variable  $k$ , is then obtained as

$$\bar{c}(A, k) = \bar{\chi}(k/A) A, \quad (49)$$

where its dependence on the initial stock of knowledge  $A$  – which remains constant with respect to optimal dynamics – has been emphasized in order to eventually reach a formulation for a value function that is a function of  $A$ . To approximate the value function  $\bar{V}(A, k)$  of problem (46) again we follow Privileggi (2013) and exploit the HJB equation as [see equation (45) in Privileggi, 2013]

$$\bar{V}(A, k) = \frac{1}{\rho} \left[ \frac{\bar{c}(A, k)^{1-\sigma} - 1}{1-\sigma} + \frac{\theta k^\alpha A^{1-\alpha} - \bar{c}(A, k)}{\bar{c}(A, k)^\sigma} \right], \quad (50)$$

where  $\bar{c}(A, k)$  is given by (49).

## 7 Simulations

We assume the following values for the fundamentals parameters in our economy, which are common in the macroeconomic literature (see, *e.g.*, Mulligan and Sala-i-Martin, 1993, and Privileggi, 2010, 2011, 2013) and satisfy both the necessary growth condition (16) and condition (48) of Proposition 4 in Privileggi (2013):<sup>7</sup>

$$\alpha = 0.5, \quad \rho = 0.04, \quad \theta = 1, \quad \sigma = 1 \text{ (log utility)}, \quad \beta = 0.01429. \quad (51)$$

Our goal is to perform comparative dynamics analysis among different optimal transition trajectories characterized by the same parameters' values as in (51) and starting from the same values of initial stock of knowledge,  $A_0$ , and physical capital,  $k_0$ , under different values of the policy parameter,  $0 \leq \tau < 1$ , which are assumed to be constant over time. Specifically, we consider the following three values:

$$\tau = 0, 0.5, 0.9. \quad (52)$$

From Proposition 4 in Marchese et al. (2014) it is known that to each  $\tau$ -value in (52) corresponds a different (transitory) turnpike  $\tilde{k}_\tau(A)$  as defined in (12), each lying one below the other for increasing values of the parameter  $\tau$ , with  $\tilde{k}_0(A)$  – *i.e.*, the turnpike corresponding to  $\tau = 0$  – on top, characterizing a scenario in which the largest capital/knowledge ratio is optimal to sustain growth, and  $\tilde{k}_{0.9}(A)$  at the bottom, envisaging an equilibrium with smaller capital/knowledge ratio and closer to the the first-best solution. Each curve converges to its own

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<sup>6</sup>See footnotes 13 and 14 in Privileggi (2013).

<sup>7</sup>The  $\beta$  value has been chosen in order to contain the error in the simulation of the optimal policy in all three IPR policy regimes considered in (52). Note that the parameter values we have chosen imply a growth rate of around 2% in (53), which is very close to that in Acemoglu and Akcigit (2012).

linear asymptotic turnpike,  $\tilde{k}_\tau^\infty(A)$ , as defined in (13), which are parallel lines corresponding to long-run balanced growth with the same constant growth rate given by (17):

$$\gamma = 0.0198. \quad (53)$$

Figure 2 depicts in the  $(A, k)$  space the stagnation line  $\hat{k}(A) = (156.25)A$  in dark grey, and three turnpike curves in *light grey*, *dark grey* and *black*, corresponding to  $\tau = 0$ ,  $\tau = 0.5$  and  $\tau = 0.9$  respectively. We shall identify with these three colors all relevant curves related to these three examples in subsequent figures. To the value for the initial stock of knowledge,  $A_0$ , common to all three IPRs policy regimes, correspond three capital values on each turnpike:  $\tilde{k}_0(A_0) > \tilde{k}_{0.5}(A_0) > \tilde{k}_{0.9}(A_0)$ . To simplify our analysis we assume that the initial condition common to all IPRs policy scenarios is the point  $(A_0, k_0) = (A_0, \tilde{k}_0(A_0))$  on the ‘highest’ turnpike among the three, corresponding to the tighter IPRs policy regime ( $\tau = 0$  under full decentralization); specifically, we set<sup>8</sup>  $A_0 = 2.2190$  and  $k_0 = 887.6010$ . Note that this choice for the  $A_0$ -value is such that the corresponding initial capital values on each turnpike lie all above the capital value  $\hat{k}(A_0) = 346.7191$  on the stagnation line (the point on the bottom left corner in Figure 2), for all  $\tau$ -values in (52); that is,  $\tilde{k}_\tau(A_0) > \hat{k}(A_0)$  for all  $\tau = 0, 0.5, 0.9$ .

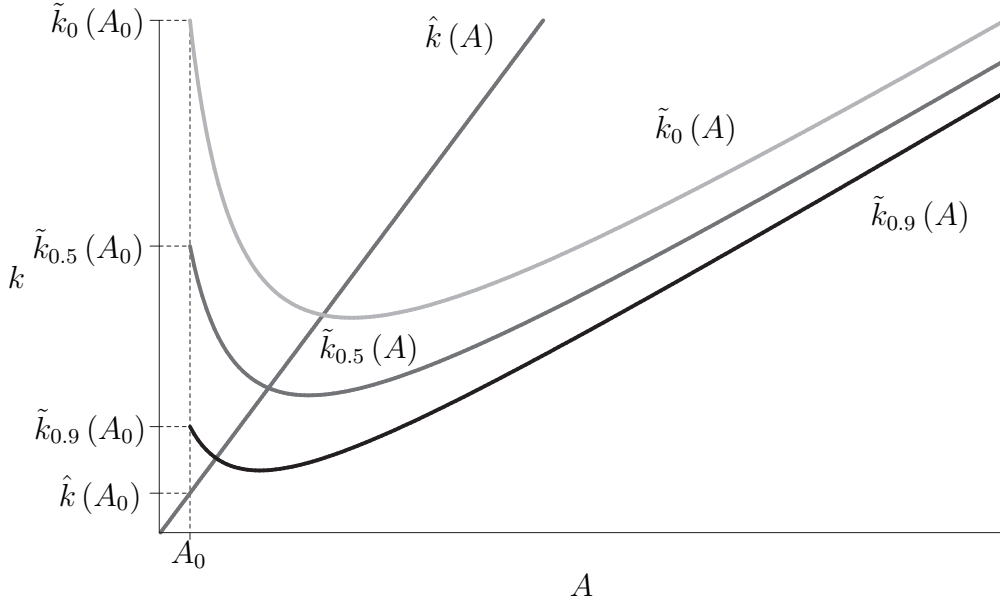


FIGURE 2: the stagnation line and three turnpikes, for  $\tau = 0, 0.5, 0.9$ .

## 7.1 Trajectories Along the Turnpikes

For the parameters' values in (51) we are able to produce satisfactory approximations of the optimal trajectories for our economy in all IPRs policy scenarios considered in (52). We exploit the value of the unique steady state  $(\mu^*, \chi^*) = (69.9790, 5.5983)$  defined in (34) plus those of all different steady states  $(\mu_\tau^s, \chi_\tau^s)$  defined in (36), each corresponding to a  $\tau$ -value in (52) and satisfying the left inequality in condition (37) for our parameters' values as in (51); specifically, for  $\tau = 0$ ,  $\tau = 0.5$  and  $\tau = 0.9$  we have  $(\mu_0^s, \chi_0^s) = (230.9069, 15.1956)$ ,  $(\mu_{0.5}^s, \chi_{0.5}^s) = (203.9620, 14.2815)$  and  $(\mu_{0.9}^s, \chi_{0.9}^s) = (181.5192, 13.4729)$ .

<sup>8</sup>See the discussion after conditions (54) and (55) below.

In order to allow the Projection algorithm to work fine in all scenarios we establish the initial stock of knowledge,  $A_0$ , by choosing a  $\mu$ -value larger than the largest value for the  $\mu_\tau^s$ s considered, corresponding to the  $\tau = 0$  (full decentralization) regime,  $\mu_0^s = 230.9069$ ; specifically, we set  $\mu_{\max} = 400$ . Using (26) with  $\tau = 0$ , we get

$$A_0 = 2/[R(\mu_{\max}) - 2] + 3/2 = 2.2190, \quad (54)$$

to which corresponds the initial capital value

$$k_0 = \tilde{k}_0(A_0) = 887.6010 \quad (55)$$

on the highest turnpike (the light grey curve in Figure 2). A large upper bound for  $\mu$ ,  $\mu_{\max} = 400$ , has been chosen because it allows each upper bound  $\bar{\mu}_\tau = \tilde{k}_\tau(A_0)/A_0$  in the Projection method to be larger than  $\mu_\tau^s$  for all  $\tau$ -values in (52). Insofar, we obtain policy simulations on the (compact) intervals  $[\underline{\mu}_\tau, \bar{\mu}_\tau]$  given by  $[\mu^*, \bar{\mu}_0] = [\mu^*, \mu_{\max}] = [69.9790, 400]$ ,  $[\mu^*, \bar{\mu}_{0.5}] = [69.9790, 283.6535]$  and  $[\mu^*, \bar{\mu}_{0.9}] = [69.9790, 190.5762]$ , all including the steady states abscissae  $\mu_0^s$ ,  $\mu_{0.5}^s$  and  $\mu_{0.9}^s$ .

Thus, the whole range of the analysis is  $[\mu_{\min}, \mu_{\max}] = [\mu^*, \bar{\mu}_0] = [69.9790, 400]$ . In all IPRs policy scenarios we set  $n = 7$  as the largest degree of the Chebyshev polynomials for the OLS-Projection method, while the integral of the squares of the residual function associated to equation (38) is approximated through Gauss-Chebyshev quadrature over 57 nodes on each interval  $[\mu^*, \bar{\mu}_\tau]$ . We apply a *constrained optimization routine* using the two equality constraints provided by the steady state pairs  $(\mu^*, \chi^*)$  and  $(\mu_\tau^s, \chi_\tau^s)$  for each  $\tau$ -regime. Specifically, we use the *Maple 16 nonlinear programming (NLP) solver* with the *sequential quadratic programming (sqp)* method, exploiting the vector given by a *Chebyshev regression of order 7* on the segment crossing the two steady states  $(\mu^*, \chi^*)$  and  $(\mu_\tau^s, \chi_\tau^s)$  as initial condition for the algorithm. See Privileggi (2011, 2013) for all caveats related to such a procedure.

Figure 3 plots in the  $(\mu, \chi)$  space the approximated optimal policies  $\chi_\tau(\mu)$  for the three examples considered,  $\chi_0(\mu)$ ,  $\chi_{0.5}(\mu)$  and  $\chi_{0.9}(\mu)$  (in light grey, dark grey and black respectively), obtained through our procedure together with loci and steady states of all three cases (the latter are the balls colored light grey to black). Accuracy tests show that our results exhibit a worst (largest) maximum error of 0.015, corresponding to the highest peak of the residual function<sup>9</sup> for the  $\tau = 0$  case. The other  $\tau$ -values yield better results, with the best (smallest) maximum error of 0.001 when  $\tau = 0.9$ . Like in similar works, residual functions do not really oscillate around zero, exhibiting a qualitative pattern similar to that in Figure 3 of Privileggi (2011), while the 8 coefficients associated to each Chebyshev polynomial of the approximation function follow a uniformly decreasing pattern at least up to the fifth coefficient, after which they start to oscillate. All in all, these results exhibit a slightly better performance than those from the analogous computations in Privileggi (2011), while they are definitely worse than those in Privileggi (2013).

According to Subsection 4.3, the analysis proceeds by numerically approximating the optimal consumption policies  $\tilde{c}_\tau(A)$  in (39) starting from  $(A_0, \tilde{k}_\tau(A_0))$  in  $t = 0$  on each turnpike and then evolving along the turnpike itself for each IPRs policy regime envisaged in (52). These policies are then employed in the HJB equation (40) to determine the value function  $\tilde{V}_\tau(A)$  of problem (18), yielding total welfare as a function of the stock of knowledge  $A \geq A_0$  when the economy evolves along each turnpike toward the ABGP. Such results will be used later either

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<sup>9</sup>Recall that the (absolute value of the) residual function tell us how far from the true policy our approximation is in each IPR policy scenario: the larger the residual, the worse the approximation.

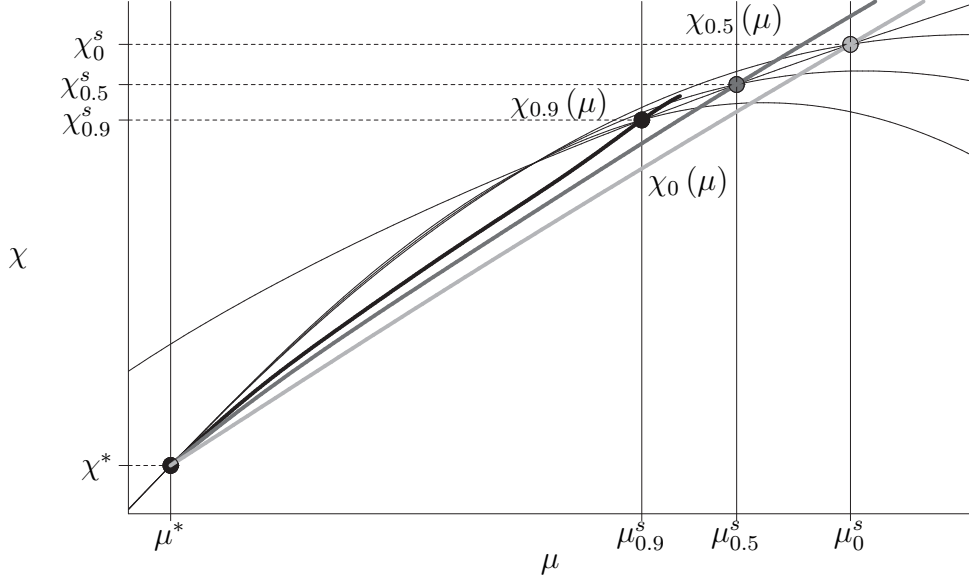


FIGURE 3: phase diagrams, steady states and approximate detrended policies for  $\tau \in \{0, 0.5, 0.9\}$ .

together with welfare estimations when the economy ends up in stagnation to evaluate Skiba-type points on each turnpike in Subsection 7.2, or, according to (45), together with welfare estimations along early trajectories outside the turnpike in Subsection 7.4.

Applying the method described at the end of Subsection 4.3 we obtain all approximate time-path trajectories: Figures 4(a) and 4(b) report  $\mu_\tau(t)$  and  $\chi_\tau(t)$  for  $0 \leq t \leq 400$  and  $\tau \in \{0, 0.5, 0.9\}$ , Figure 4(c) draws the  $\tilde{A}_\tau(t)$  trajectories all starting from  $A_0 = 2.2190$  in  $t = 0$ , while Figures 4(d), 4(e) and 4(f) show the corresponding optimal capital,  $\tilde{k}_\tau(t)$ , output,  $\tilde{y}_\tau(t)$ , and consumption,  $\tilde{c}_\tau(t)$ , trajectories.

Note that, because here our goal was to approximate the optimal time-path trajectories along each turnpike independently of other conditions, we have assumed that  $t_0 = 0$  in all three scenarios. Therefore, in Figure 4(d) the initial capital values in  $t = 0$  are all different from each other, as they correspond to their own turnpike value on the same initial stock of knowledge  $A_0 = 2.2190$ :  $\tilde{k}_0(A_0) = k_0 = 887.6010 > \tilde{k}_{0.5}(A_0) > \tilde{k}_{0.9}(A_0)$  (see Figure 2). Similarly, also the initial output and consumption values in Figures 4(e) and 4(f) are different. Hence, such trajectories do not provide the correct information aimed at performing comparative dynamics, as, unlike assuming identical initial conditions as in the following Subsections 7.3 and 7.4, they start from different initial capital endowments.

## 7.2 Skiba-Points On the Turnpikes

Our next goal is to approximate the function  $\bar{c}(A, k)$  according to (49) in Section 6 yielding the optimal policy toward the stagnation point  $(A, \hat{k}(A)) = (A, (156.25)A)$  on the stagnation line, starting from any initial pair  $(A, k)$  such that  $A \geq A_0$  and with  $k/A$  ratio values in the range<sup>10</sup>  $[\hat{k}(A)/A, k_0/A_0]$ , that is, as  $k_0/A_0 = 887.6010/2.2190 = 400$ , for  $A \geq A_0$  and  $k/A \in [156.25, 400]$ . Recall that in all such computations  $A$  is being kept constant; also

<sup>10</sup>This range is justified by the assumption that all initial capital values considered in our simulation lie above the steady state value  $\hat{k}(A) = (156.25)A$  on the stagnation line and are equal or less than the upper bound  $k_0 = 887.6010$ .



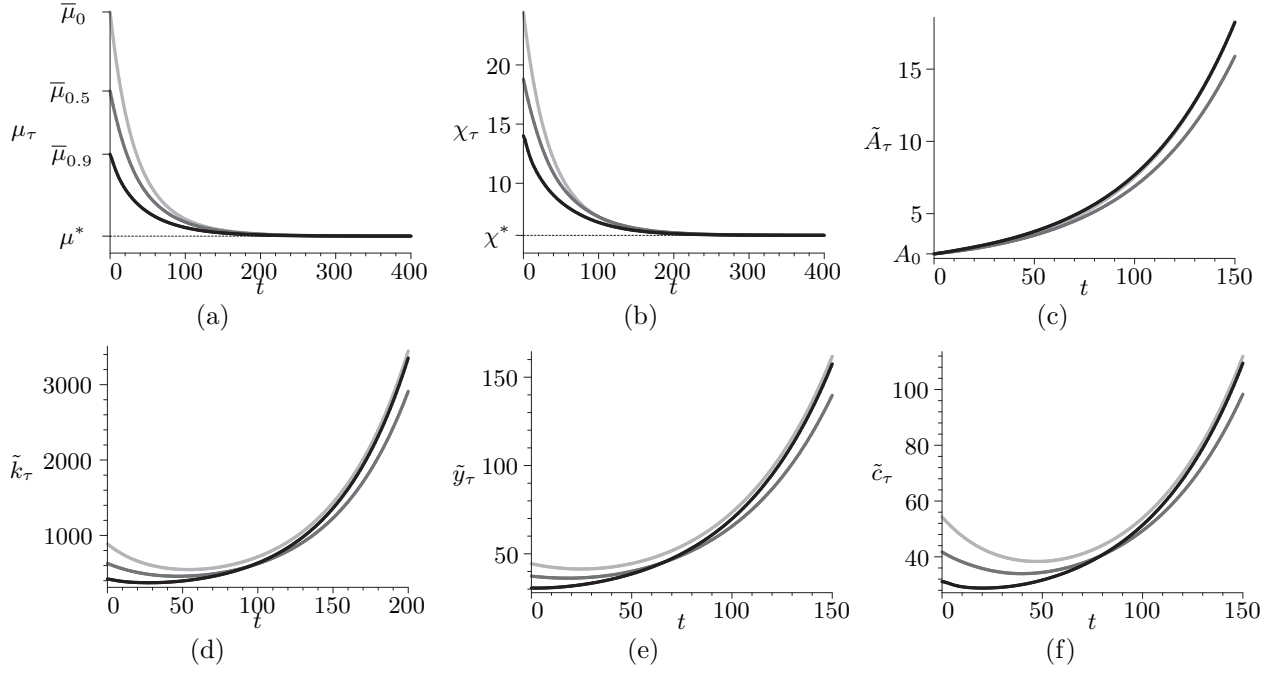


FIGURE 4: time-path trajectories of (a)  $\mu_\tau(t)$ , (b)  $\chi_\tau(t)$ , (c)  $\tilde{A}_\tau(t)$ , (d)  $\tilde{k}_\tau(t)$ , (e)  $\tilde{y}_\tau(t)$  and (f)  $\tilde{c}_\tau(t)$  along the turnpikes, for  $\tau = 0$  (light grey),  $\tau = 0.5$  (dark grey) and  $\tau = 0.9$  (black).

note that this routine can be run just once for all IPRs policy regimes, as the optimal policy toward stagnation, in its detrended variables version, turns out to be the same for all scenarios. Hence, following Section 6, we numerically approximate the solution of the ODE (48) through a Collocation-Projection method based on  $n = 14$  collocation points over the (compact) interval  $[\hat{\mu}, \mu_{\max}] = [156.25, 400]$ , where  $\hat{\mu} = \hat{k}(A)/A$  is defined in (35) while  $\mu_{\max} = k_0/A_0$  is the upper-bound already chosen in Subsection 7.1. Such a choice for the range  $[\hat{\mu}, \mu_{\max}]$  – specifically, for the left endpoint  $\hat{\mu}$  – is motivated by our assumptions that the trajectories in all scenarios here considered start from points above the stagnation line, implying that  $\mu(0) > \hat{k}(A)/A$  always hold. The resulting approximated solution of (48), that we label  $\bar{\chi}(\mu)$ , exhibits an outstanding performance with a maximum error of  $10^{-9}$  and a residual function symmetrically oscillating around zero, while the 15 coefficients associated to each Chebyshev polynomial of the approximation function uniformly decrease from  $a_0 = 19.9320$  to  $a_{14} = 8.7 \times 10^{-10}$ .

Using the function  $\bar{\chi}(\mu)$  just obtained in (49) we get the optimal policy  $\bar{c}(A, k)$  that, in turn, allows for the approximation of the value function  $\bar{V}(A, k)$  of problem (46), yielding total welfare when the economy evolves toward stagnation as a function of any initial stock of knowledge  $A \geq A_0$  and initial capital  $k$  such that  $k/A \in [156.25, 400]$ , through the HJB equation (50).

With both value functions  $\tilde{V}_\tau(A)$  and  $\bar{V}(A, k)$  obtained so far, we can exploit the techniques developed by Privileggi (2013) in order to evaluate the unique Skiba-point on each turnpike corresponding to the values of  $\tau$  considered in (52). Specifically, because our choice of parameters' values in (51) satisfies condition (48) of Proposition 4 in Privileggi (2013), after reformulating each turnpike according to (12) and using the value functions  $\tilde{V}_\tau(A)$  and  $\bar{V}(A, k)$  just calculated according to (40) and (50) respectively, we apply the analogous of Algorithm 1 in Privileggi (2013) to approximate the unique  $(\tilde{A}_\tau^{sk}, \tilde{k}_\tau^{sk}) = (\tilde{A}_\tau^{sk}, \tilde{k}_\tau(\tilde{A}_\tau^{sk}))$  pair that satisfies  $\tilde{V}_\tau(A) = \bar{V}[A, \tilde{k}_\tau(A)]$ , *i.e.*, that equates the social welfare yield by the trajectory starting on

$(\tilde{A}_\tau^{sk}, \tilde{k}_\tau^{sk})$  and then following its optimal path along the turnpike toward the ABGP to the welfare generated by the optimal path leading to stagnation.

These findings are listed in Table 1: it is remarkable the decreasing pattern of both the initial stock of knowledge,  $\tilde{A}_\tau^{sk}$ , and stock of capital,  $\tilde{k}_\tau^{sk}$ , required to allow the economy to take off toward sustained growth when starting on the turnpike, with respect to increasing  $\tau$ -values. In other words, larger values of parameter  $\tau$  relieve the initial conditions required for the economy to grow in the long-run, allowing poorer countries to undertake a growth path. It is also noticeable from the second column in Table 1 that, as  $\tilde{A}_\tau^{sk} > A_0 = 2.2190$  for all  $\tau$ -values in (52), if the economy were supposed to start with an initial capital endowment corresponding to  $\tilde{k}_\tau(A_0)$  on each turnpike, none of the scenarios considered in our study would allow a sustained growth pattern under our choice of initial stock of knowledge  $A_0 = 2.2190$ , as the social planner would find optimal to lead the economy to eventual stagnation, regardless the IPRs policy regime chosen.

$\tau$	$\tilde{A}_\tau^{sk}$	$\tilde{k}_\tau^{sk}$
0	3.0149	581.4472
0.5	2.7154	488.8517
0.9	2.3462	400.5483

TABLE 1: Skiba-points in terms of initial knowledge,  $\tilde{A}_\tau^{sk}$ , and capital,  $\tilde{k}_\tau^{sk}$ , on each turnpike.

### 7.3 Trajectories that Start Above the Turnpikes

To complete our study, from now on we assume that the economy starts from the same initial condition on the highest turnpike, that is on the pair  $(A_0, k_0) = (2.2190, 887.6010)$  – corresponding to the left endpoint on the light grey curve on the top left corner of Figure 2 – and perform a truly comparative dynamics analysis across the different values of the IPRs policy parameter  $\tau$  listed in (52). As the value  $k_0 = \tilde{k}_0(A_0) = 887.6010$  lies on the turnpike defined by the value  $\tau = 0$ , only for the last two  $\tau$ -values,  $\tau = 0.5, 0.9$ , we must consider an early period of time, from  $t = 0$  to  $t = t_0 > 0$ , during which the optimal time-path trajectories follow the pattern discussed in Section 5 starting from the initial condition  $(A_0, k_0)$  – which lies strictly ‘above’ all values  $\tilde{k}_\tau(A_0)$  for  $\tau > 0$  – before entering their own turnpike at  $t = t_0$  and keep growing along it thereafter.

Running the bisection Algorithm 1 presented in Section 5 for  $\tau = 0.5, 0.9$  we get two values for the stock of knowledge,  $A_\tau^r > A_0$ , which are reported in the second column of Table 2, and two optimal policies  $k_\tau^{ab}(A)$  and  $c_\tau^{ab}(A)$ . We used the the last two Skiba values  $\tilde{A}_\tau^{sk}$  in the second column of Table 1 as starting point for the search of the right endpoint of the initial bracketing interval  $[A_0, A_R]$  for the zero of  $f$  in step 1 of Algorithm 1. We actually ran Algorithm 1 for several  $\tau$ -values other than 0.5 and 0.9. Depending on the  $\tau$ -value, we set  $n$  (the number of nodes and the largest degree of the Chebyshev polynomials in the Collocation-Projection method) between 21 for smaller  $\tau$  and 24 for larger  $\tau$ ; similarly the step increment  $\epsilon$  has been set equal to 1 for smaller  $\tau$  and equal to 2 for larger  $\tau$ , while the stopping rules has been set at  $\varepsilon = \eta = 10^{-7}$  for all  $\tau$ -values. All policies estimations exhibit quite reasonable maximum errors of order  $10^{-6}$ , corresponding to the highest peak of the residual functions for the  $k_\tau^{ab}(A)$  policies [maximum errors for the  $c_\tau^{ab}(A)$  policies are on average  $10^{-2}$  smaller], residual functions symmetrically oscillating around zero and all  $n + 1$  coefficients associated to each Chebyshev polynomial of the approximation function uniformly decreasing for all  $\tau$ -values.

The time elapsed for running Algorithm 1 varies from 158 seconds when  $\tau = 0.3$  to 229 seconds when  $\tau = 0.8$  on an Intel Dual-Core CPU machine with 4GB RAM, with numbers of iterations ranging from 27 to 29.

Figure 5(a) shows the functions  $k_{0.5}^{ab}(A)$  and  $k_{0.9}^{ab}(A)$  representing the optimal capital associated to the stock of knowledge for  $A_0 \leq A \leq A_{0.5}^r$  and  $A_0 \leq A \leq A_{0.9}^r$ , respectively, before entering the turnpikes  $\tilde{k}_{0.5}(A)$  and  $\tilde{k}_{0.9}(A)$ , according to the last two rows in the second column of Table 2, at the points  $A_{0.5}^r = 5.2342$  and  $A_{0.9}^r = 6.1187$  (thick part of the curves).<sup>11</sup> Similarly, Figure 5(b) reports the functions  $c_{0.5}^{ab}(A)$  and  $c_{0.9}^{ab}(A)$  defining the optimal consumption associated to the stock of knowledge for  $A_0 \leq A \leq A_{0.5}^r$  and  $A_0 \leq A \leq A_{0.9}^r$  respectively, before entering their turnpike consumption policies  $\tilde{c}_{0.5}(A)$  and  $\tilde{c}_{0.9}(A)$  as computed in Subsection 7.1, at the points  $A_{0.5}^r$  and  $A_{0.9}^r$  (thick part of the curves). Note the kinks at the points  $A_{0.5}^r$  in both figures.

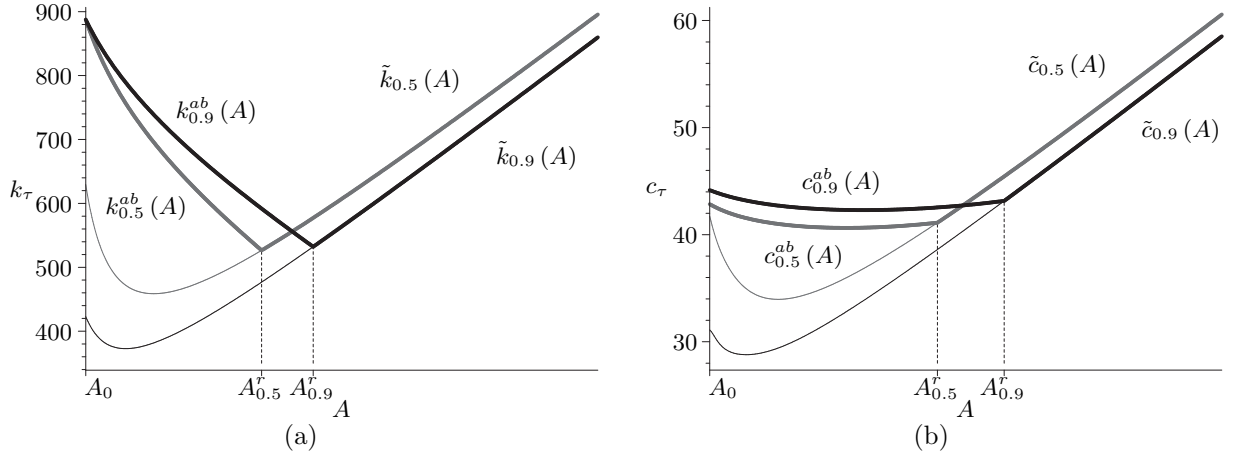


FIGURE 5: functions (a)  $k_\tau(A)$  and (b)  $c_\tau(A)$  representing optimal capital and consumption before hitting their turnpike values at  $A_{0.5}^r$ , after which they become turnpike themselves, for  $\tau = 0.5$  (dark grey) and  $\tau = 0.9$  (black).

Following the steps described after Algorithm 1 in Section 5, we first numerically compute the optimal time-path trajectories  $A_\tau^{ab}(t)$ ,  $k_\tau^{ab}(t)$ ,  $y_\tau^{ab}(t)$  and  $c_\tau^{ab}(t)$  between  $t = 0$  and some instant  $t_{\max} > 0$ , the latter to be determined through some guess-and-try, being careful to choose a value larger than the (still unknown) instant  $t_0$  at which each turnpike is reached. A value of  $t_{\max}$  around 10 worked well in all our simulations. Next, by solving  $A_\tau^{ab}(t) = A_r$  with respect to  $t$  over  $[0, t_{\max}]$ , the corresponding instants  $t_0 > 0$  at which the optimal trajectories from above intersect their respective turnpikes are found; they are reported in the last two rows of the third column of Table 2 [in the first row  $A_0^r = A_0$  and  $t_0 = 0$  because, for  $\tau = 0$ , the initial knowledge-capital pair lies exactly on the highest turnpike,  $\tilde{k}_0(A_0)$ ]. Finally, the whole optimal time-path trajectories  $A_\tau(t)$ ,  $k_\tau(t)$ ,  $y_\tau(t)$ ,  $c_\tau(t)$  and  $G_\tau(t)$  over  $(0, +\infty)$  when the economy starts at  $t = 0$  from  $(A_0, k_0) = (2.2190, 887.6010)$  are obtained by joining the trajectories above the turnpike over  $[0, t_0]$  just calculated with their ‘continuation’ along the turnpike over  $(t_0, +\infty)$  provided by the analysis of Subsection 7.1. Figure 6 plots  $A_\tau(t)$ ,  $k_\tau(t)$ ,  $y_\tau(t)$  and  $c_\tau(t)$ : the light grey curves, corresponding to the zero IPRs policy parameter regime  $\tau = 0$ , denote trajectories starting already on the  $\tilde{k}_0(A)$  turnpike at  $t = 0$  which keep moving along it thereafter, while the dark grey and black curves, corresponding to the positive IPRs

<sup>11</sup>The whole turnpikes  $\tilde{k}_{0.5}(A)$  and  $\tilde{k}_{0.9}(A)$  for  $A \geq A_0$  are the union of the thin curves on the left with the thick ones to the right in the figure, corresponding to the dark grey and black curves in Figure 2.

policy parameter regimes  $\tau = 0.5$  and  $\tau = 0.9$ , are the piecewise trajectories obtained by joining<sup>12</sup> at  $t = t_0$  the trajectories above the turnpikes for  $0 \leq t \leq t_0$  with their continuation along the turnpikes for  $t \geq t_0$ .

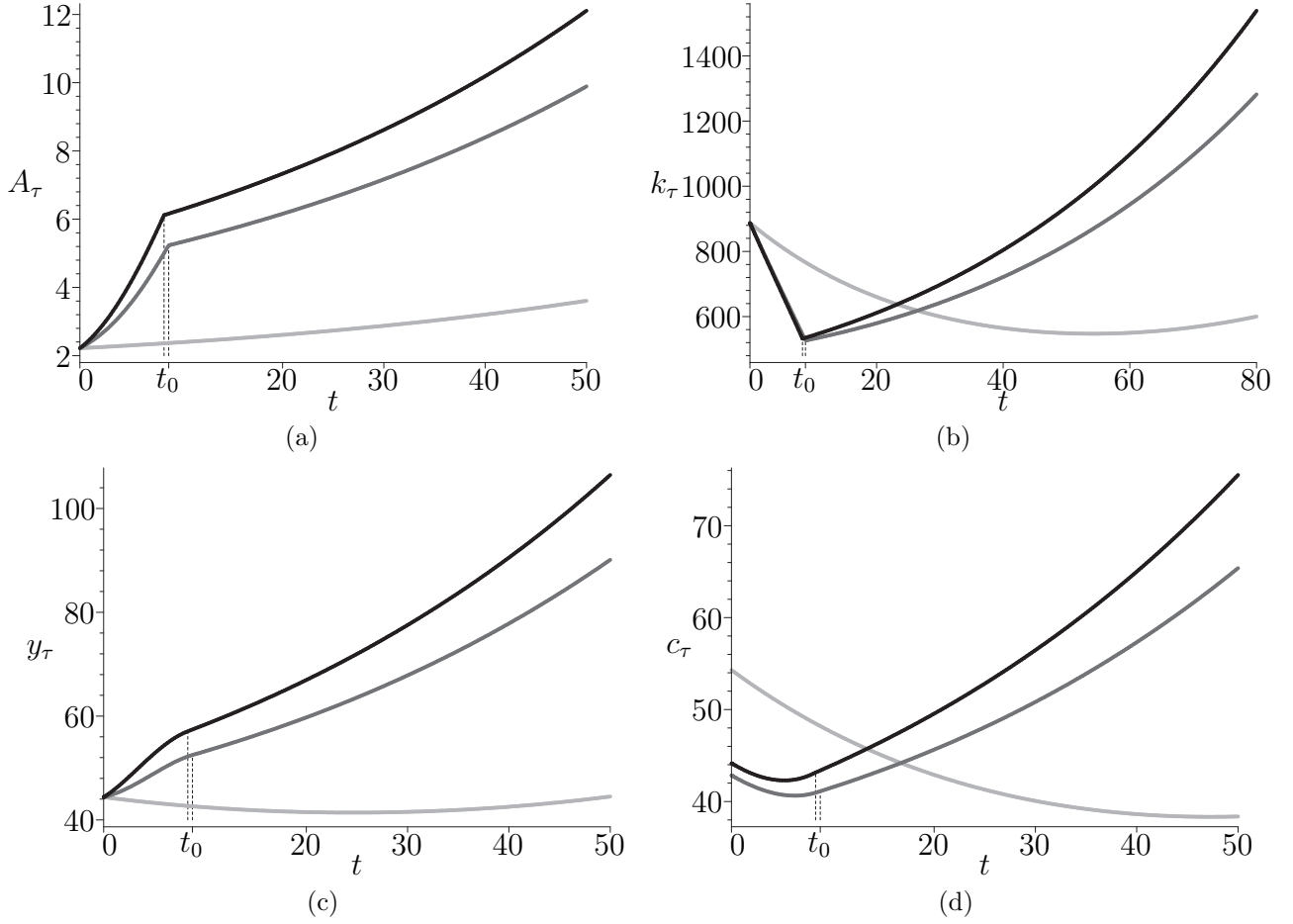


FIGURE 6: whole optimal transition time-path trajectories, (a)  $A_\tau(t)$ , (b)  $k_\tau(t)$ , (c)  $y_\tau(t)$  and (d)  $c_\tau(t)$ , under the three tax-subsidy regimes,  $\tau = 0$  (full decentralization) in light grey,  $\tau = 0.5$  in dark grey and  $\tau = 0.9$  in black, all starting from  $A_0 = 2.2190$  and  $k_0 = 887.6010$  in  $t = 0$ .

It turns out that along the time-path trajectories  $\tilde{A}_\tau(t)$ ,  $\tilde{k}_\tau(t)$ ,  $\tilde{y}_\tau(t)$  and  $\tilde{c}_\tau(t)$  along each turnpike as computed in Subsection 7.1 for  $\tau = 0.5, 0.9$ , if the economy were to start already on the turnpike at  $t = 0$  (rather than from the capital level  $k_0 = 887.6010$  strictly above each turnpike), it would take around ten times longer than  $t_0$  to reach the same intersection points  $A_\tau^r$ . In other words, the time period  $t_0$  required to enter the turnpikes *starting from above the turnpikes* is around *ten times shorter* than the time period needed to reach the same points when *starting already on the turnpikes*. Indeed, an initial capital endowment  $k_0$  which is strictly larger than the  $\tilde{k}_\tau(A_0)$  level on the turnpike is beneficial in that it implies that it is optimal to invest the maximum amount allowed in the production of new knowledge, which equals output,  $G(t) = y(t)$ , a policy that translates into steeper  $A(t)$ ,  $y(t)$  trajectories for  $0 \leq t \leq t_0$  [dark grey and black curves in Figures 6(a) and 6(c)], indicating a high boost to growth provided during the initial paths covered ‘above’ the turnpikes. Such initial boost, in turn, explains the better performances of all  $A_\tau(t)$ ,  $k_\tau(t)$ ,  $y_\tau(t)$  and  $c_\tau(t)$  trajectories under

<sup>12</sup>From the third column in Table 2 we see that all intersection points are reached at instants  $t_0$  that are very close to each other in different IPR policy regimes.

the  $\tau > 0$  regimes (*e.g.*, those in dark grey and black in Figure 6) than their counterparts for  $\tau = 0$  (in light grey in Figure 6) when  $t$  becomes large.

Note the kinks of all the trajectories for  $\tau = 0.5$  and  $\tau = 0.9$  at the instant  $t_0$  in Figure 6 [especially  $A_\tau(t)$ ,  $k_\tau(t)$  in Figures 6(a) and 6(b)], corresponding to the kinks at the points  $A_\tau^r$  in Figure 5: on  $t_0$  such trajectories are not differentiable due to the discontinuity jump of the optimal investment in R&D activities time-path trajectory,  $G_\tau(t)$ , as it is evident from Figure 7.

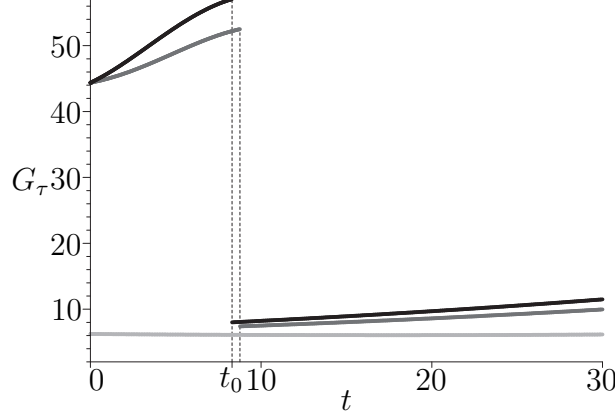


FIGURE 7: whole transition time-path trajectories of optimal R&D financing,  $G_\tau(t)$ , under the three tax-subsidy regimes,  $\tau = 0$  (full decentralization) in light grey,  $\tau = 0.5$  in dark grey and  $\tau = 0.9$  in black, all starting from  $A_0 = 2.2190$  and  $k_0 = 887.6010$  in  $t = 0$ .

From figure 6(d), showing the whole time-path trajectories of optimal consumption in the three IPRs policy parameter scenarios, however, it is not clear what the effect of the different policies on welfare may be, as in early times the  $\tau = 0.5$  and  $\tau = 0.9$  regimes envisage a lower consumption than in the  $\tau = 0$  regime, only to catch up and rapidly overcome later on. As a matter of fact, the former trajectories may fail to deliver a welfare higher than that yield by the latter if discounting assigns less weight to later consumption than that assigned to early consumption.

## 7.4 Welfare Estimates

The last two columns of Table 2 report our main welfare results. For the initial condition  $(A_0, k_0) = (2.2190, 887.6010)$  common to all  $\tau$ -values in (52), the social welfare estimates in the fifth column of Table 2 are computed through (45) at the end of Section 5, where, whenever  $t_0 > 0$ , the first integral on the RHS is approximated through a *Gauss-Legendre quadrature* routine on 1000 nodes over each time range  $[0, t_0]$ , using the time-path trajectories approximations of optimal consumption obtained in the previous subsection,  $c_\tau^{ab}(t)$ , on each node. Note that the first scenario, corresponding to  $\tau = 0$ , implies  $t_0 = 0$  so that the social welfare along the turnpike and toward the ABGP would actually be given by (40) at the end of Subsection 4.3, yielding  $\tilde{V}_0^{ab}(A_0, A_0) = \tilde{V}_0(A_0) = 95.2964$ . However, from Table 1 we know that when  $\tau = 0$  and  $(A_0, k_0) = (2.2190, 887.6010)$  the economy does not satisfy the Skiba condition; this is confirmed by the first value in the fourth column of Table 2, and implies that the first number in the fifth column comes from the value function toward stagnation estimated through (50) at the end of Section 6,  $\bar{V}(A_0, k_0) = \bar{V}(2.2190, 887.6010) = 95.8745$ , rather than the former smaller estimate  $\tilde{V}_0(A_0) = 95.2964$ . The other two welfare values in the fifth column, estimated through the value function  $\tilde{V}_\tau^{ab}(A_0, A_\tau^r)$  as properly defined in (45), are larger than  $\bar{V}(A_0, k_0) = 95.8745$ ;

consistently,  $(A_0, k_0) = (2.2190, 887.6010)$  satisfies the Skiba condition when  $\tau = 0.5, 0.9$ , as confirmed by the last two rows in the fourth column.

The last two rows of Table 2 clearly indicate that the initial push deriving from starting above any turnpike, if on one hand entails sacrificing consumption in early times as shown by Figure 6(d), on the other hand it turns out to be sufficiently strong to 1) let the economy escape the stagnation trap otherwise forecasted by Table 1 for all regimes, including the active-policies characterized by  $\tau > 0$ , and 2) yield a social welfare that is strictly increasing in the  $\tau$  parameter values, as it is apparent from the fifth column of Table 2 and is confirmed by all other simulations we have run. Indeed, consistently with the first row in Table 1, when  $\tau = 0$  our economy happens to be born on the highest turnpike  $\tilde{k}_0(A_0)$  and doomed to renounce growth in the long-run as the Skiba condition is not satisfied. Conversely, positive IPRs policy regimes implemented by policymakers may let the same economy take off toward long-run sustained growth, as shown in the last two rows of the fourth column of Table 2. A more thorough discussion on the robustness of the results implied by Table 2 and their economic interpretation can be found in Marchese et al. (2014).

$\tau$	$A_\tau^r$	$t_0$	Toward	Social welfare
0	2.2190 ( $= A_0$ )	0	stagnation	95.8745
0.5	5.2342	8.76	ABGP	97.8152
0.9	6.1187	8.30	ABGP	99.8471

TABLE 2: Intersection points  $A_\tau^r$  between each trajectory from above and its turnpike, instants  $t_0$  at which the intersection occurs, type of equilibrium (toward either stagnation or sustained growth along an ABGP) and social welfare for the common initial condition  $(A_0, k_0) = (2.2190, 887.6010)$  in all IPR policy regimes.

## 8 Concluding Remarks

In this paper we provide a numerical method to assess the impact of different IPRs policy regimes on social welfare in the extended continuous time endogenous recombinant growth model presented in Marchese et al. (2014). The wide range of techniques used include Projection methods, Gauss-Chebyshev and Gauss-Legendre quadrature, and standard Runge-Kutta type algorithms. Thus, this work represents a further step forward in the analysis of the transitional dynamics of recombinant growth models (Privileggi, 2010, 2011, 2013) providing some interesting insights into the evaluation of alternative policies to promote economic growth and ultimately improvements in social welfare.

In order to quantitatively assess welfare effects through a comparative dynamic analysis, we need to compute the entire transitional dynamics of the optimal consumption associated to different IPRs policy regimes. This requires to distinguish between trajectories that starts on the turnpike and trajectories starting outside the turnpike, further distinguished between those converging to the turnpike and those ending up in stagnation. We develop an algorithm (based on a bisection routine) to identify the intersection point between paths starting above the turnpike and the turnpike itself in order to build the whole optimal consumption path as a piecewise function of time by joining each trajectory above the turnpike with its continuation along the turnpike. This allows to compute social welfare and thus compare the welfare levels associated to different policy parameters, for a better understanding of which IPRs policy regime might be more desirable for our society as a whole. Our simulations based on a certain

parametrization of the model allows us to conclude that welfare increases with the IPRs policy parameter, thus softer IPRs policy regimes should be preferred in order to maximize social welfare.

## Supplementary Material

The Maple 16 code for all numeric procedures described in this paper, including the detailed code for Algorithm 1 as well as for the computations performed in section 7, is available from the authors upon request.

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